Exam: Lotka-Volterra Dynamics, 2010

London Taught Course Centre

Consider the Lotka-Volterra system

$$\begin{aligned}\dot{x}_1 &= x_1(r_1 + w_{11}x_1 + w_{12}x_2 + w_{13}x_3)\\ \dot{x}_2 &= x_2(r_2 + w_{21}x_1 + w_{22}x_2 + w_{23}x_3)\\ \dot{x}_3 &= x_3(r_3 + w_{31}x_1 + w_{32}x_2 + w_{33}x_3)\end{aligned}$$

where $w_{ij} \in \mathbb{R}$ and $r_i \in \mathbb{R}$. Suppose that $w_{ij} \neq 0$ if $i \neq j$.

Explore the conditions on the r_i and the matrix W with elements w_{ij} for which this system is Hamiltonian and find a suitable Poisson bracket and Hamiltonian function. Are there any Casimirs?

Suppose that $r_1 = -1, r_2 = 1, r_3 = 0$ and $w_{12} = -w_{21} = 1, w_{31} = -w_{13} = 1, w_{23} = -w_{32} = 1$. Explore the possible existence of periodic orbits.

Show that any planar periodic orbit of the above system encloses a convex set with strictly convex boundary. Is this also true for any planar system with quadratic vector field? (Hint: how many times may a line touch the boundary of a planar convex set?)