LTCC

Variational and Computational Methods for PDEs

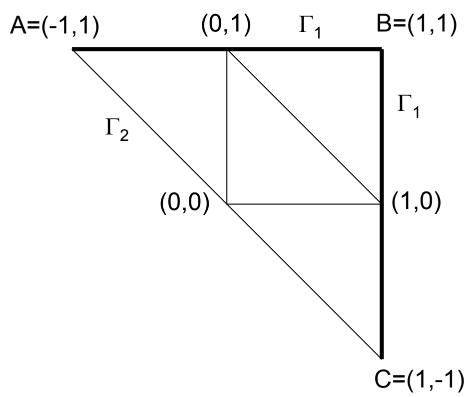
Exam Question, Spring 2010

Let Ω be a triangle with the vertices at points $A = (-1, 1)$, $B = (1, 1)$, $C = (1, -1)$ in $\mathbb{R}^2$, see Figure. Consider the boundary value problem

$$-\mathrm{div}\,[(1 + x_1 + x_2)^{\alpha}\,\nabla u(x)] + \beta(1 + x_2)u(x) = f(x), \quad x \in \Omega,$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_2, \qquad\qquad u = 0 \quad \text{on } \Gamma_1,$$

where $\alpha, \beta \geq 0$ are some constants, $f \in L_2(\Omega)$, $g \in L_2(\Gamma_2)$,
Γ_1 is the line ABC, Γ_2 is the line AC.



1. Derive a variational formulation for this problem.
2. Prove that the variational formulation has a unique solution in an appropriate space.
3. Give the general finite-element formulation of the problem.
4. Give the dimensionality and the basis functions of the piece-wise linear finite element space on the four-element mesh shown on the figure, for this problem.
5. Calculate a finite element piece-wise linear approximation of the solution on the four-element mesh for $\alpha = \beta = 0$, $f(x) = 1$, $g(x) = 1$ and calculate $u(1/2, 1/2)$.