LTCC Course on Stochastic Processes
Question for examination 2010.

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ALL parts of this question should be attempted.

(a) Suppose that a Markov chain has state space $S = \{1, 2, 3, 4\}$ and transition matrix $\boldsymbol{P}$ given by

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ a & 0 & 1-a & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ b & 0 & 1-b & 0 \end{pmatrix}$$

where $0 < a < 1$ and $0 < b < 1$.

Show that the state space forms a single irreducible class of states. For what values of a and b is the class ergodic?

If $a = b = 1/4$, is this Markov chain reversible when it is in equilibrium?

More generally, for what values of a and b is this Markov chain reversible when it is in equilibrium?

(b) In a retail warehouse, shoppers first queue to order and pay for their goods, and then join a second queue to collect their shopping. At each of the two stages, they form a single queue and are served in order of arrival by one of a pair of servers. The customers arrive at the first service point in a Poisson process of rate 16 per hour. The service times of customers at the first stage are exponentially distributed with a mean of 6 minutes, while those at the second stage are exponentially distributed with a mean of 5 minutes. You may assume that all service times are mutually independent and are independent of the arrival process.

Let $X_1(t)$ be the total number of customers queueing to *order* their goods at time t and let $X_2(t)$ be the total number queueing to *collect* them, in each case including any customers currently being served.

Explain why an equilibrium distribution exists for $\{X_1(t), X_2(t)\}$.

Assuming that the warehouse has been open sufficiently long that the system can be assumed to be in equilibrium, explain why $\{X_1(t)\}$ is a reversible process, and find its equilibrium distribution.

Describe the process of customers arriving at the second stage to collect their goods and find the equilibrium distribution for $\{X_2(t)\}$.

Now consider the complete system $\{X_1(t), X_2(t)\}$. Is this process reversible? By using the appropriate balance equations, show that the joint equilibrium distribution is given by the product of the two marginal equilibrium distributions. Does this mean that $\{X_1(t)\}$ and $\{X_2(t)\}$ are independent (give an explanation for your answer).

(c) A nonhomogeneous Poisson process, $\{N(t)\}$, has an intensity function, $\rho(t)$, that alternates between two constant values. For the 'odd' intervals $2m < t \leq 2m+1$, $\rho(t) = \rho_1$, while for the 'even' intervals $2m+1 < t \leq 2m+2$, $\rho(t) = \rho_2$, where $m = 0, 1, \ldots$.

Suppose that the starting point of an interval of unit length is chosen 'at random' (*i.e.* has a continuous distribution that is uniform over a very long time period). Derive the probability generating function of the number of events in this interval, together with its mean.

Now suppose that a new origin, O, uniformly distributed on the interval $(0, 2]$, is defined, and the Poisson process on $(O, \infty]$ is considered relative to that origin. Show that this process is stationary with a constant intensity function. Comment on the connection between this result and the distribution you obtained above.

Finally, suppose that a doubly stochastic Poisson process is defined by taking the alternating rates ρ_1 and ρ_2 as applying in alternate intervals of a renewal process, starting with rate ρ_1 in the first interval. What can you say about the intensity of this process? How does this simplify if the renewal process is a Poisson process of rate λ.

(d) Consider an SI model for an infection spreading in a population of size n. Each infective makes potentially infectious contacts at rate λ, choosing the person contacted at random (for mathematical convenience 'self-contacts' are possible); if this person is susceptible then they instantly become infected. The infection is permanent, with no immunity or recovery, so that once infected the individual remains infective indefinitely. Let $I(t)$ denote the number of infectives at time t, and assume that process starts with a single infective at $t = 0$.

Write down the embedded Markov chain for this process and use it to describe the distribution of the time until the infection has spread through the whole population. Give an expression for the mean of this duration and evaluate it for $n = 4$.

Now consider an SIS model in which the infection process is as before, but individuals have independent, exponentially distributed infectious periods (all with mean $1/\gamma$) before recovering and returning to the susceptible state. Write down the embedded Markov chain for this process and use it to explain why the infection will eventually become extinct. Write down a set of simultaneous equations for the mean durations of the epidemic, if the initial number of infectives were i for $i = 0, 1, 2, \ldots, n$, and discuss their solution. Find the duration of the epidemic explicitly starting with a single infective, for the case $n = 3$.