London Taught Course Centre

2010 examination

Graph Theory

Instructions to candidates

This part of the exam has one question, consisting of several parts. You are expected to **answer all parts**.

Justify all your answers.

1 Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function. A graph G is f*-sparse* if, for every subset X of $V(G)$, the number of edges in the induced subgraph $G[X]$ on X is at most $f(|X|)$.

For instance, take $f(x) = x - 1$. Then a graph G is f-sparse if and only if there are at most $x - 1$ edges between any set of x vertices. This means that G cannot have any cycles, and so for this function, G is f-sparse if and only if G is a forest.

(a) Show that, if G is an f-sparse graph on n vertices, then

$$\chi(G) \leq \max_{t \leq n} \frac{2f(t)}{t} + 1.$$

(b) For $f(x) = x^2/3$, show that a graph G is f-sparse if and only if it does not contain a copy of K_4 (as a subgraph).

(c) Show that if $f(x) = x + C$ for some constant C, then f-sparsity is closed under taking minors.

Give an example of a function f for which f-sparsity is not closed under taking minors.

(d) Show that every planar graph with girth (length of the shortest cycle) g is f-sparse, for $f(x) = \dfrac{g}{g-2}(x-2)$.

(e) For fixed $n \in \mathbb{N}$ and $p \in (0, 1)$, set

$$f(x) = f_{n,p}(x) = \frac{px^2}{2} + x^{3/2}\sqrt{\frac{p \log n}{2}} + \frac{x \log n}{3}.$$

Let $G_{m,p}$ be a random graph on m vertices. Show that the probability that $G_{m,p}$ has more than $f_{n,p}(m)$ edges is at most n^{-m}.

Find an absolute constant $\delta > 0$ such that the probability that a random graph $G_{n,p}$ is not $f_{n,p}$-sparse is at least δ, for any n and p.

You may wish to use the *Chernoff bound*: for a binomial random variable Z with mean μ, and any $t > 0$,

$$\Pr(Z \geq \mu + t) \leq \exp\left(-\frac{t^2}{\mu + t/3}\right).$$