p-adic numbers, LTCC 2010

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Exam, April 2010

Let p be a prime number.

Definition. A function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is called *locally constant* if every point $x \in \mathbb{Z}_p$ has a neighbourhood U such that the restriction of f to U is a constant function. The set of all locally constant functions $\mathbb{Z}_p \to \mathbb{Q}_p$ will be denoted by $\mathrm{LC}(\mathbb{Z}_p, \mathbb{Q}_p)$.

Definition. Let $c > 0$. A function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ *satisfies a Lipschitz condition of order* c if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M \cdot |x - y|^c$$

for all $x, y \in \mathbb{Z}_p$. The set of all functions $\mathbb{Z}_p \to \mathbb{Q}_p$ satisfying a Lipschitz condition of order c will be denoted by $\mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$.

Question. Show that there are strict inclusions

$$\mathrm{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \subset \bigcap_{c>0} \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p) \qquad \text{and} \qquad \bigcup_{c>0} \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p) \subset C(\mathbb{Z}_p, \mathbb{Q}_p)$$

where $C(\mathbb{Z}_p, \mathbb{Q}_p)$ is the set of all continuous functions $\mathbb{Z}_p \to \mathbb{Q}_p$. You can (but you don't have to) proceed in the following steps.

(a) Show that $\bigcup_{c>0} \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p) \subseteq C(\mathbb{Z}_p, \mathbb{Q}_p)$, i.e. show that if $f \in \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$ for some $c > 0$ then f is continuous.

(b) Show that $\mathrm{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \subseteq \bigcap_{c>0} \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$, i.e. show that if $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is a locally constant function then $f \in \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$ for all $c > 0$.

(c) Show that $\mathrm{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \neq \bigcap_{c>0} \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$ as follows. Define a function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ by

$$f(a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \dots) = a_0 + a_1 p^{1!} + a_2 p^{2!} + a_3 p^{3!} + \dots$$

where $a_0, a_1, a_2, a_3, \dots \in \{0, 1, 2, \dots, p-1\}$. Show that if $|x - y| = p^{-k}$ then $|f(x) - f(y)| = p^{-k!}$, and deduce that $f \in \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$ for all $c > 0$. Show that f is not locally constant.

(d) Show that $\bigcup_{c>0} \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p) \neq C(\mathbb{Z}_p, \mathbb{Q}_p)$ by constructing a continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ such that $f \notin \mathrm{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$ for all $c > 0$.