LTCC Applied Bayesian Methods
2009 Exam Question

Question I

Reliable records are available for the dates of all earthquakes in England since 1850. Let T_i denote the time between the i and $(i+1)$th earthquake ($i = 1, 2, \ldots$). $T_1, T_2, \ldots$ are independently identically distributed exponential with rate θ.

1. Derive the Jeffreys prior for θ and show that this corresponds to assuming a uniform prior distribution on $\log \theta$. Briefly explain the main advantage and disadvantages of using Jeffreys prior as a non-informative prior.
2. Over the 150 years between 1850 and 2000 there were 126 earthquakes in England. The mean time, $\bar{T}$, between these earthquakes was 429 days. Using these data and the prior derived in part 1, obtain the posterior distribution for θ.
3. Using a normal approximation to this posterior distribution, calculate an approximate 95% posterior credible interval for θ.
4. Derive the predictive distribution (up to a constant of proportionality) for T_{126}, the time between the last recorded earthquake and the next one, and show that this corresponds to a Gamma-Gamma distribution. What are the parameters of this Gamma-Gamma distribution?

Note: the probability density function of a Gamma-Gamma distribution is as follows. If $X \sim$ Gamma-Gamma(α, β, τ), then

$$p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha + \tau)}{\Gamma(\tau)} \frac{x^{\tau-1}}{(\beta + x)^{\alpha+\tau}} \quad (x, \alpha, \beta > 0;\ \tau = 1, 2, \ldots)$$

Question II

Consider the following model. For $i = 1, \ldots, n$ and $j = 1, 2, 3, 4$,

$$\begin{aligned} Y_{ij} \mid \alpha_i, \beta_i &\sim \text{Normal}(\alpha_i - \beta_i \times j^{-1}\ ,\ \tau^{-1}) \\ \alpha_i \mid \mu_\alpha, \tau_\alpha &\sim \text{Normal}(\mu_\alpha, \tau_\alpha^{-1}) \\ \beta_i \mid \mu_\beta, \tau_\beta &\sim \text{Normal}(\mu_\beta, \tau_\beta^{-1}) \end{aligned}$$

The following prior distributions are specified for μ_α, μ_β, τ_α, τ_β and τ:

$$\begin{aligned} \mu_\alpha &\sim \text{Normal}(0, 10^4) \\ \mu_\beta &\sim \text{Normal}(0, 10^4) \\ \tau_\alpha &\sim \text{Gamma}(0.1, 0.1) \\ \tau_\beta &\sim \text{Gamma}(0.1, 0.1) \\ \tau &\sim \text{Gamma}(1, 1) \end{aligned}$$

1. Draw a Directed Acyclic Graph for this model.
2. State the Factorisation Theorem and use it to derive the joint distribution of all the random variables in the model.
3. Draw the conditional independence graph that corresponds to the Directed Acyclic Graph in part (a). Write down the Markov blanket of each of Y_{ij}, α_i and τ_β.
4. Which of the following statements are true? Explain your answers.
   (a) $\alpha_i \perp\!\!\!\perp \beta_i$
   (b) $\alpha_i \perp\!\!\!\perp \beta_i \mid Y_{ij}$
   (c) $Y_{i1} \perp\!\!\!\perp Y_{i2} \mid \alpha_i, \beta_i$
5. Suppose you have worked out the full conditional distributions of α_i, β_i, μ_α, μ_β, τ_α, τ_β and τ. Explain in detail how these are used in the Gibbs sampler algorithm to obtain a estimate of the posterior variance of α_i.
6. The method known as 'batching' or the 'method of batched means' is used to measure what quantity? Why is it important to measure this quantity? (Note: it is not necessary to explain how the method works.)
7. Explain how you could use an empirical Bayes approach to avoid the need to specify the priors for μ_α, μ_β, τ_α and τ_β. What are the disadvantages of this approach?

Question III

A team of alien psycho-surgeon researchers is at work in London, removing the brains of Londoners and replacing them with cauliflowers. For two reasons, the local authorities wish to stop this research. Firstly, no ethics committee has granted permission for it, and secondly, the tax-payer is incurring considerable expense cleaning up after the aliens. Each time a Londoner is operated on by the aliens there is a certain probability that his or her head will explode during the operation, creating a terrible mess.

While the London Committee for Extraterrestrial Research Control Measures tries to agree a date for a meeting to discuss the drawing up a prevention strategy policy document, you have the task of monitoring the scale of the problem. Your job is to estimate, for each week, the number of Londoners whose brains have been operated upon by the aliens in that week. For this purpose you have only the number of reported exploding heads in each week since the aliens arrived and the advice of your mathematical modellers.

It is now three weeks since the aliens arrived. In the first week, two heads exploded; in the second week, three heads; and in the third week, ten heads. These are your data. Your mathematical modellers confidently inform you of the following facts:

- The number of Londoners operated upon by the aliens in any given week has a Poisson distribution.

- The rate parameter of this Poisson distribution will be different in each week, because the alien psychosurgeons are becoming more experienced and so can carry out more operations. For any given week, with probability 0.5, the rate parameter for that week will be two times the rate parameter for the previous week; otherwise it will be three times the rate parameter for the previous week. The increases in the rate parameter in different weeks are independent.
- The probability that an operation results in the head exploding is 0.4, and the outcomes of different operations are independent.

1. Using the information above, devise a Bayesian model for this problem. This model, once fitted, should allow you to produce 95% posterior credible intervals for both the expected number of operations in each week and the actual number of operations in each week. You should define all the random variables in your model and draw a Directed Acyclic Graph showing how they relate to one another. Also, write down the likelihood equation for each of the random variables except the one that requires a prior distribution.
2. Which of the random variables in the model is the one requiring a prior distribution? Suggest a possible non-informative prior distribution for this random variable.
3. Although the mathematical modellers have told you that the probability that a head explodes when operated upon is 0.4, they admit that they do not know exactly what this probability is. Instead, they suggest that you elaborate your Bayesian model to treat this probability as a random variable π. Draw a Directed Acyclic Graph for this elaborated Bayesian model.
4. The modellers say that they are 95% certain that π lies between 0.25 and 0.55. Specify a suitable prior distribution for π.