London Taught Course Centre

2009 examination

Graph Theory

Instructions to candidates

This exam has **three questions**. You are expected to attempt all questions.

Justify all your answers.

1 Following Notes 1, an *orientation* of a graph G is an assignment of one of the two possible directions to each edge of G. A *path* in an oriented graph is always assumed to be directed, i.e., has all edges going in the same direction. The *length* of a path is the number of edges in it. An orientation is *acyclic* if it has no directed cycles.

Prove that the following two statements are equivalent for any graph G and positive integer k:

(a) G has a vertex colouring with k colours;

(b) G has an acyclic orientation without paths of length k.

2 Show that there is a constant $c > 0$ such that, for all sufficiently large n, there is a graph G on n vertices with the following properties:

(i) G has no independent set of size at least $n^{1/2}$;

(ii) for every pair $\{x, y\}$ of vertices, the number of vertices z adjacent to both x and y is at most $c \log^2 n$.

You should make c as small as you can.

[You may find the *Chernoff bounds* to be of use – see the answers to the exercises of Notes 4.]

3 Consider the following minor variant on the definition of an ε-regular pair:

Given a graph $G = (V, E)$, and $\varepsilon > 0$, a pair (V_1, V_2) of disjoint subsets of V is called *ε-regular* if, for every $A \subseteq V_1$, $B \subseteq V_2$, with $|A| \geq 2\varepsilon|V_1|$, $|B| \geq 2\varepsilon|V_2|$, we have:

$$\left| \frac{e(A,B)}{|A|\,|B|} - \frac{e(V_1,V_2)}{|V_1|\,|V_2|} \right| < \varepsilon.$$

Here $e(X, Y)$ denotes the number *of edges with one endpoint in X and the other in Y. The only difference between this and the usual definition is that we only consider sets A and B containing at least a fraction 2ε of the vertices in the respective sets.*

Suppose that (V_1, V_2) is a $\frac{1}{5}$-regular pair, with $|V_1| = |V_2| = 3$. Show that, for any $A, A' \subset V_1$, $B, B' \subset V_2$, with $|A| = |A'| = |B| = |B'| = 2$,

$$|e(A, B) - e(A', B')| \leq 1.$$

Show that, if there is some $d \in \{0, 1, 2, 3\}$ such that every vertex in V_1 is adjacent to exactly d vertices in V_2, and every vertex in V_2 is adjacent to exactly d vertices in V_1, then (V_1, V_2) is a $\frac{1}{5}$-regular pair.

Find a $\frac{1}{5}$-regular partition of the Petersen graph into sets V_0, V_1, V_2, V_3.

See Definition 3.2 of Notes 5 for the definition of an ε-regular partition. Here, an ε-regular pair is understood to be as defined in this question.