LTCC Course on Stochastic Processes
Questions for examination 2009.

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1. Consider a single server queue in which arrivals occur in a Poisson process of rate λ ($\lambda > 0$). The service times of customers are independent of the arrival sequence, and are independent, exponentially distributed random variables all with parameter μ ($\mu > 0$). However, customers get impatient and do not always wait for service. A customer (i, say) who arrives to find the server busy is prepared to wait for a random time Z_i, and will leave and not return if service has not begun by then. Assume that the Z_i are independent and identically distributed random variables and are independent of the rest of the queueing system. Let $X(t)$ be the number of customers in the system at time t.

   (a) Explain whether or not $\{X(t)\}$ is, in general, a Markov process.

   (b) Suppose, for the rest of this question, that the Z_i are exponentially distributed with parameter γ ($\gamma > 0$). Does this affect your answer above?

   (c) Explain whether or not $\{X(t)\}$ is an M/M/1 queue (you may find it helpful to consider its generator).

   (d) Intuitively, for what conditions on λ, μ and γ, if any, would you expect $\{X(t)\}$ to possess an equilibrium distribution?

   (e) Assuming that $\{X(t)\}$ is stationary, explain why it is a reversible process. Find its equilibrium distribution and confirm (or not) your intuitive answer in part (d) above.

   (f) What can you say about the sequence of customers departing the system?

2. Events occur in a renewal process, with interval density $f_X(x) = \lambda^2 x e^{-\lambda x}$ $(x > 0)$ (*i.e.* the intervals have a gamma distribution $\Gamma(2, \lambda)$).

   (a) Each event, independently of the rest, is deleted with probability p. Explain why the resulting point process is a renewal process, and find the corresponding renewal density. Determine the mean number of retained events in an interval of length τ that starts from a retained event.

   (b) Alternatively, each interval of the renewal process is deleted with probability p, so that the events at either end of a deleted interval become coincident, and the resulting process has multiple occurrences. For this resulting point process, state the joint distribution of the intervals between distinct event times and of the multiplicities of the events at those times. For this process, determine the mean number of events (including the multiplicities) in an interval $(0, \tau]$ that starts from an event at 0.

   (c) In each case, specify (*i.e.* name) the processes that would result from each of these operations if the original renewal process were a Poisson process of rate λ rather than having gamma-distributed intervals.

3. Consider the following very simple Markov process, sometimes used to model the initial stages of an epidemic when the infection has a very long infectious period. An infection is introduced (at time $t = 0$) into a susceptible group of size $n-1$ by a single infected individual. Let $I(t)$ be the number of those who have been infected by time t, including the initial infective, so that $I(0) = 1$. In this model, $I(t)$ increases at a rate $\alpha I(t)[1 - I(t)/n]$ until it reaches n and then remains constant, as all infected individuals remain infected indefinitely. In the application, only relatively small values of t are of interest.

   (a) Derive the forward equation for the moment generating function $M(\theta; t) = \mathrm{E}(e^{-\theta U(t)})$ of $U(t) = I(t)/n$.

   (b) By expanding this equation in powers of $1/n$ as described in Section 4.2 of the notes, show that $U(t)$ is asymptotically normally distributed as $n \to \infty$, and write down the differential equations for its mean and variance.