p-adic numbers, LTCC 2010

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Let $(K, |\ |)$ be a complete non-archimedean valued field, and let X be a non-empty compact topological space. We denote the set of continuous functions $X \to K$ by $C(X, K)$. For a continuous function $f : X \to K$ we define $\|f\| = \sup_{x \in X} |f(x)|$.

Definition. A function $f : X \to K$ is called *locally constant* if every point $x \in X$ has a neighbourhood U such that the restriction of f to U is a constant function.

Question.

(a) Do Exercise 3.17, i.e. show that $(C(X, K), \|\ \|)$ is a Banach space over K.

(b) Let $\mathrm{LC}(X, K)$ be the set of all locally constant functions $X \to K$. Show that $\mathrm{LC}(X, K)$ is a dense subspace of $C(X, K)$.

(c) Now let p be a prime number and let K be a complete extension of $\mathbb{Q}_p$. Let $a \in K$ satisfy $|a - 1| < 1$, so that we can define a continuous function $\mathbb{Z}_p \to K$, $x \mapsto a^x = \sum_{n=0}^{\infty} (a-1)^n \binom{x}{n}$, as in §4.3. Show that this function is locally constant if and only if a is a p-power root of unity, i.e. $a^{p^n} = 1$ for some $n \in \mathbb{N}$.