

The Tutte Polynomial

Steve Noble

27/11/2010

Research at Brunel

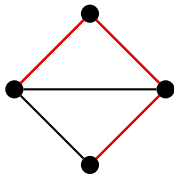
- Brunel is in Uxbridge, as far west in London as it is possible to go!
- We have been a university since 1966.
- 45 research students, 30 full-time academic staff;
- 3 fully funded PhD bursaries per year. University wide Isambard scholarships.

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 - applied analysis;
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 - continuum mechanics;
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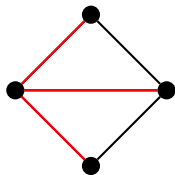
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- Please see our brochure or our webpage at www.brunel.ac.uk/siscm/mathematical-sciences.html

Counting Spanning Trees



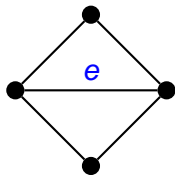
$\tau(G)$

Counting Spanning Trees

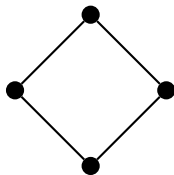


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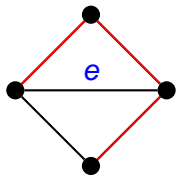


$\tau(G - e)$

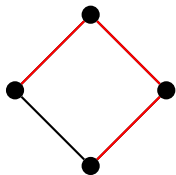


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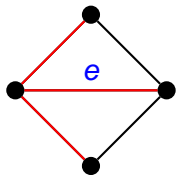


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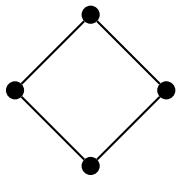


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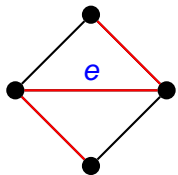


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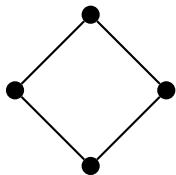


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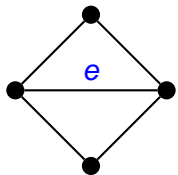


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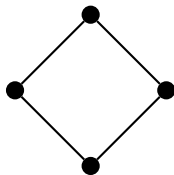
$\tau(G/e)$

Counting Spanning Trees



$\tau(G)$

=



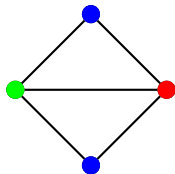
$\tau(G - e)$

+



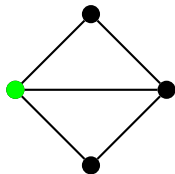
$\tau(G/e)$

Chromatic Polynomial



$$\chi(G; \lambda)$$

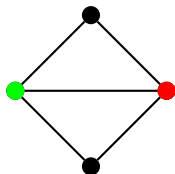
Chromatic Polynomial



$$\chi(G; \lambda)$$

$$\chi(G; \lambda) = \lambda$$

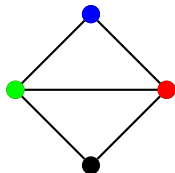
Chromatic Polynomial



$$\chi(G; \lambda)$$

$$\chi(G; \lambda) = \lambda(\lambda - 1) \quad .$$

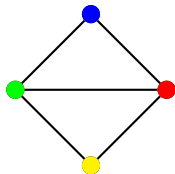
Chromatic Polynomial



$$\chi(G; \lambda)$$

$$\chi(G; \lambda) = \lambda(\lambda - 1)(\lambda - 2) .$$

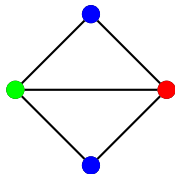
Chromatic Polynomial



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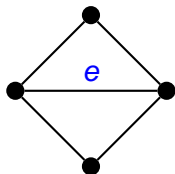
Chromatic Polynomial



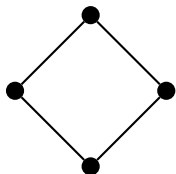
$\chi(G; \lambda)$

$$\chi(G; \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2.$$

Chromatic Polynomial



$$\chi(G; \lambda)$$

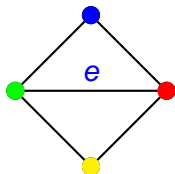


$$\chi(G - e; \lambda)$$

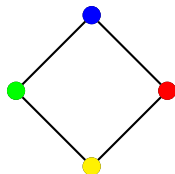


$$\chi(G/e; \lambda)$$

Chromatic Polynomial



$$\chi(G; \lambda)$$

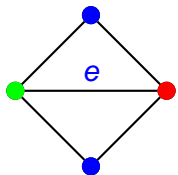


$$\chi(G - e; \lambda)$$

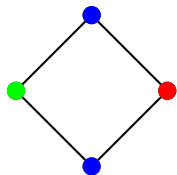


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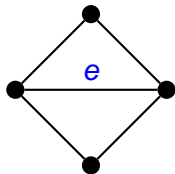


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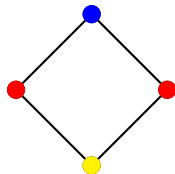


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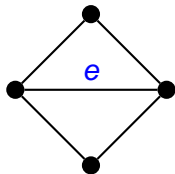


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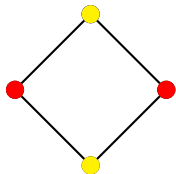


$$\chi(G/e; \lambda)$$

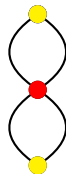
Chromatic Polynomial



$$\chi(G; \lambda)$$

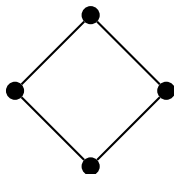
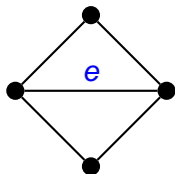


$$\chi(G - e; \lambda)$$



$$\chi(G/e; \lambda)$$

Chromatic Polynomial



$$\chi(G; \lambda) = \chi(G - e; \lambda) - \chi(G/e; \lambda)$$

The Tutte Polynomial

Definition (Tutte, 1948)

The Tutte polynomial T is defined as follows:

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- If G is a one edge graph and the edge is a loop then $T(G; x, y) = y$.

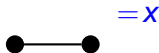


The Tutte Polynomial

Definition (Tutte, 1948)

The Tutte polynomial T is defined as follows:

- $T(G; x, y)$ is multiplicative over connected components / blocks;
- If G is a one edge graph and the edge is a loop then $T(G; x, y) = y$.
- If G is a one edge graph and the edge is a bridge then $T(G; x, y) = x$.



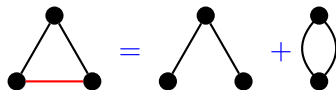
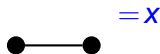
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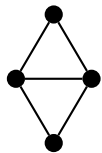
The Tutte polynomial T is defined as follows:

- $T(G; x, y)$ is multiplicative over connected components / blocks;
- If G is a one edge graph and the edge is a loop then $T(G; x, y) = y$.
- If G is a one edge graph and the edge is a bridge then $T(G; x, y) = x$.
- If e is an edge of G that is neither a loop nor a bridge then

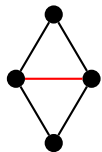
$$T(G; x, y) = T(G - e; x, y) + T(G/e; x, y).$$



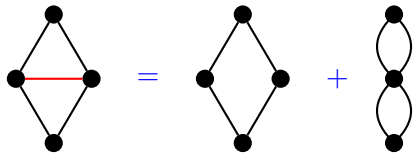
An Example



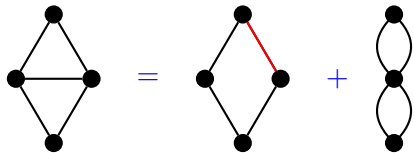
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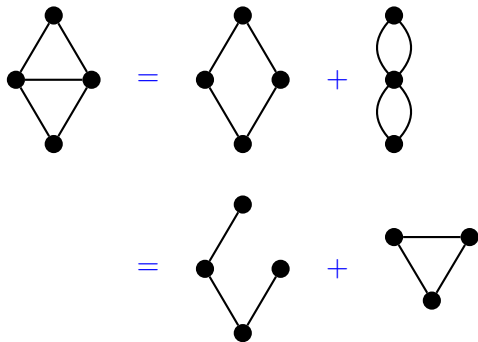
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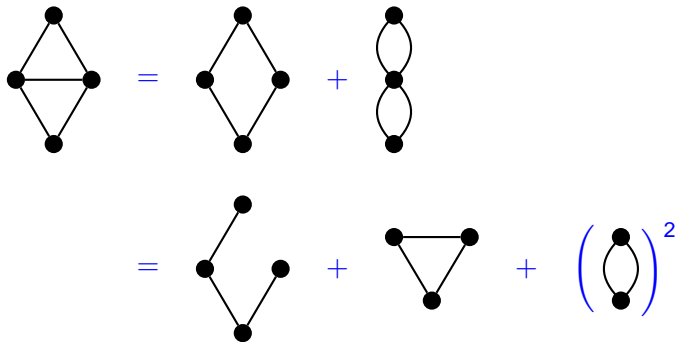
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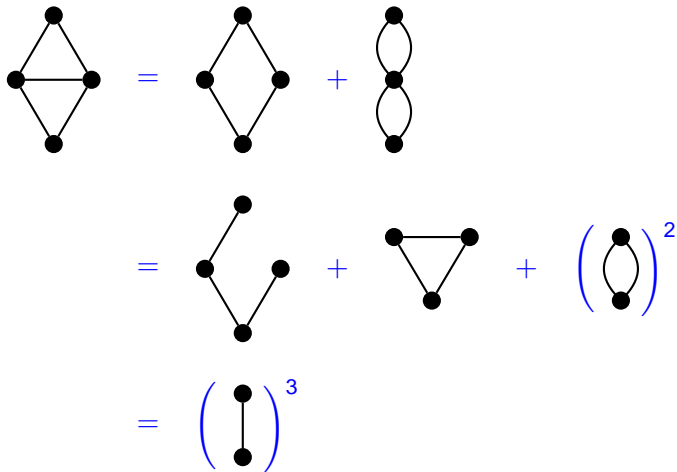
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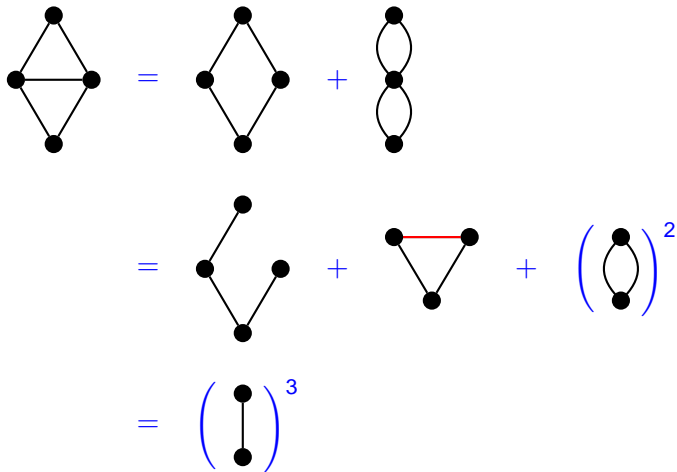
An Example



An Example



An Example



An Example

The diagram illustrates the decomposition of a graph with four vertices and five edges (a diamond shape with a horizontal edge) into simpler components. The decomposition is shown in three rows:

- Row 1: The diamond graph with a horizontal edge is equal to the diamond graph without the horizontal edge plus two parallel edges between the top and bottom vertices.
- Row 2: The diamond graph without the horizontal edge is equal to a path of three edges plus a triangle plus the square of the two parallel edges.
- Row 3: The path of three edges is equal to the cube of the two parallel edges plus a path of two edges plus the two parallel edges.

An Example

The diagram illustrates the decomposition of a graph with four vertices and five edges (a diamond shape with a horizontal edge) into a sum of simpler graphs. The decomposition is shown in three rows:

- Row 1: The diamond graph with a horizontal edge is equal to the diamond graph without the horizontal edge plus two parallel edges between the top and bottom vertices.
- Row 2: The diamond graph without the horizontal edge is equal to a path of three edges plus a triangle plus the square of a graph with two vertices and two edges (one black, one red).
- Row 3: The square of the two-vertex graph is equal to the cube of a graph with two vertices and one edge plus a V-shaped graph plus a graph with two vertices and two edges.

An Example

$$\begin{aligned} \text{Graph 1} &= \text{Graph 2} + \text{Graph 3} \\ &= \text{Graph 4} + \text{Graph 5} + \left(\text{Graph 6} \right)^2 \\ &= \left(\text{Graph 7} \right)^3 + \text{Graph 8} + \text{Graph 9} + \left(\text{Graph 10} + \text{Graph 11} \right)^2 \end{aligned}$$

The diagram illustrates the decomposition of a graph into a sum of other graphs. The graphs are defined as follows:

- Graph 1:** A diamond-shaped graph with four vertices and five edges (forming a K4 minus one edge).
- Graph 2:** A diamond-shaped graph with four vertices and four edges (forming a cycle).
- Graph 3:** Two separate edges, each with two vertices.
- Graph 4:** A path of three edges connecting four vertices.
- Graph 5:** A triangle with three vertices and three edges.
- Graph 6:** Two separate edges, each with two vertices.
- Graph 7:** A single edge with two vertices.
- Graph 8:** A path of two edges connecting three vertices.
- Graph 9:** A single edge with two vertices.
- Graph 10:** A single edge with two vertices.
- Graph 11:** A loop with one vertex and one edge.

An Example

$$\begin{aligned} & \text{A diamond graph with a horizontal edge} = \text{A diamond graph without a horizontal edge} + \text{Two vertically stacked loops} \\ & = \text{A path of length 3} + \text{A triangle} + \left(\text{Two vertically stacked loops} \right)^2 \\ & = \left(\text{Two vertically stacked loops} \right)^3 + \text{A path of length 2} + \text{Two vertically stacked loops} + \left(\text{Two vertically stacked loops} + \text{A loop} \right)^2 \\ & = x^3 \end{aligned}$$

An Example

$$\begin{aligned}
 & \text{[A diamond graph with a horizontal edge]} = \text{[A diamond graph without a horizontal edge]} + \text{[Two parallel edges between two vertices]} \\
 & = \text{[A path of length 3]} + \text{[A triangle]} + \left(\text{[Two parallel edges]} \right)^2 \\
 & = \left(\text{[Two parallel edges]} \right)^3 + \text{[A V-shaped graph]} + \text{[Two parallel edges]} + \left(\text{[Two parallel edges]} + \text{[A loop]} \right)^2 \\
 & = x^3 + x^2 + \text{[Two parallel edges]} + \left(\text{[Two parallel edges]} + \text{[A loop]} \right)^2
 \end{aligned}$$

An Example

$$\begin{aligned}
 & \text{[A diamond graph with a horizontal edge]} = \text{[A diamond graph without a horizontal edge]} + \text{[Two vertical edges with loops]} \\
 & = \text{[A path of length 3]} + \text{[A triangle]} + \left(\text{[Two vertical edges with loops]} \right)^2 \\
 & = \left(\text{[Two vertical edges with loops]} \right)^3 + \text{[A V-shape]} + \text{[Two vertical edges with loops]} + \left(\text{[Two vertical edges with loops]} + \text{[A loop]} \right)^2 \\
 & = x^3 + x^2 + (x + y) + (x + y)^2
 \end{aligned}$$

Some Evaluations

- $T(G; 1, 1)$ is the number of spanning trees of a connected graph G .

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- $T(G; 1, 1)$ is the number of spanning trees of a connected graph G .
- $\lambda^{k(G)}(-1)^{|V|+k(G)} T(G; 1 - \lambda, 0)$ is the chromatic polynomial of G .

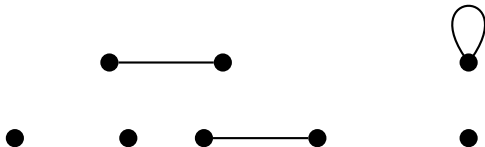
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- $T(G; 1, 1)$ is the number of spanning trees of a connected graph G .
- $\lambda^{k(G)}(-1)^{|V|+k(G)}T(G; 1 - \lambda, 0)$ is the chromatic polynomial of G .
- $T(G; 2, 1)$ is the number of forests and the number of score vectors of G .



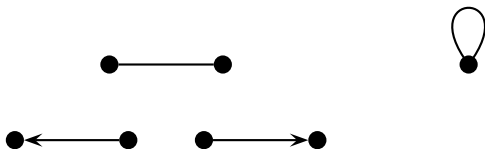
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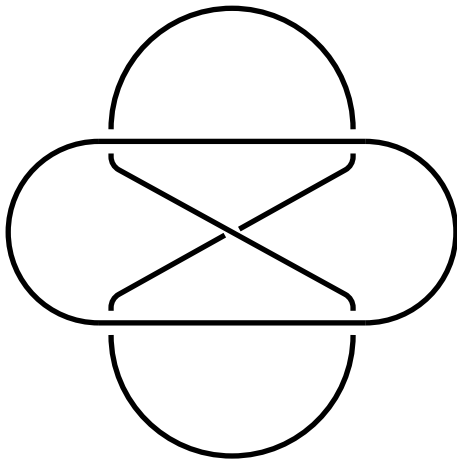
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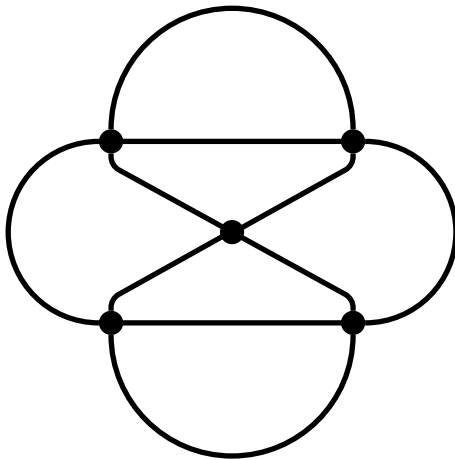
- $T(G; 1, 1)$ is the number of spanning trees of a connected graph G .
- $\lambda^{k(G)}(-1)^{|V|+k(G)}T(G; 1 - \lambda, 0)$ is the chromatic polynomial of G .
- $T(G; 2, 1)$ is the number of forests and the number of score vectors of G .
- $T(G; 2, 0)$ is the number of acyclic orientations of G .

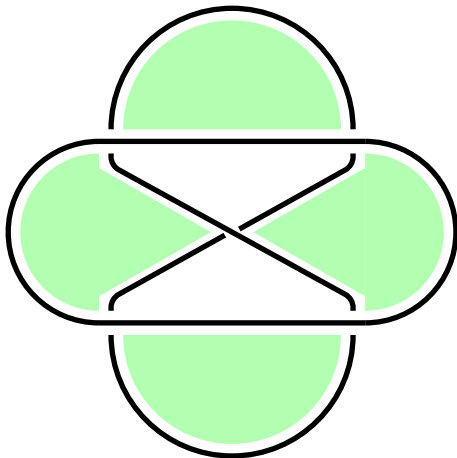


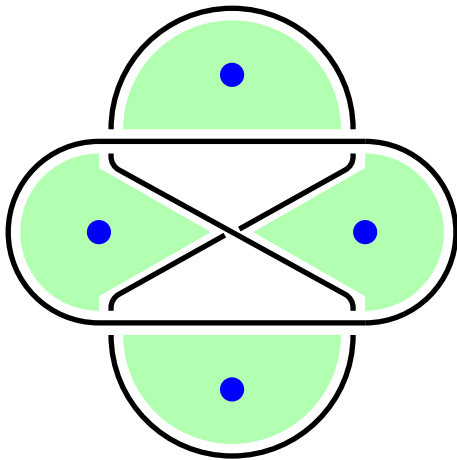
Some Evaluations

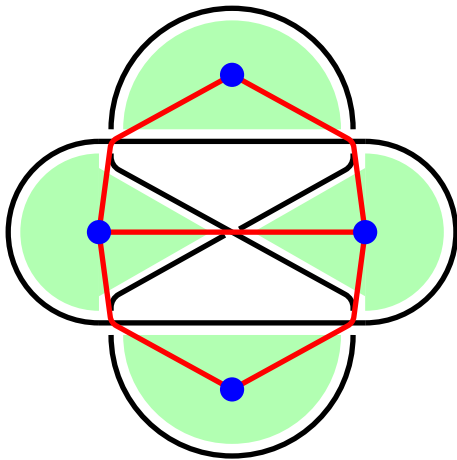
- $T(G; 1, 1)$ is the number of spanning trees of a connected graph G .
- $\lambda^{k(G)}(-1)^{|V|+k(G)}T(G; 1 - \lambda, 0)$ is the chromatic polynomial of G .
- $T(G; 2, 1)$ is the number of forests and the number of score vectors of G .
- $T(G; 2, 0)$ is the number of acyclic orientations of G .
- Other specializations to applications from operational research, statistical physics and topology.

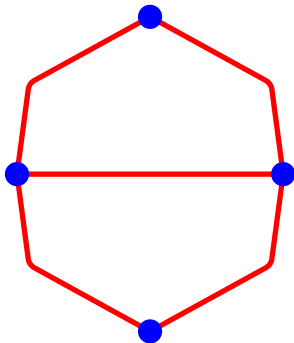


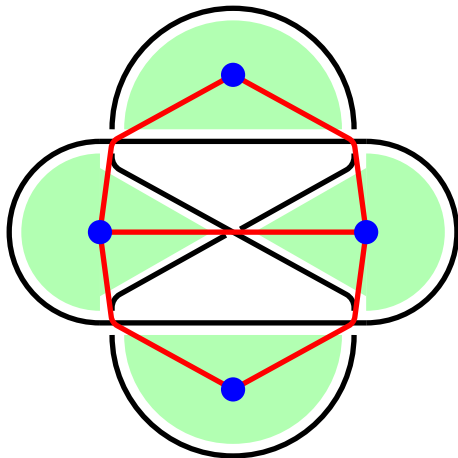








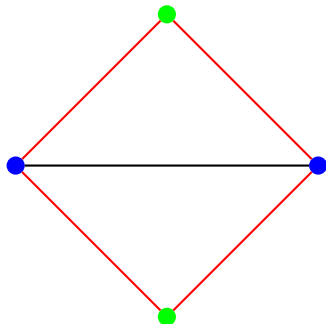




Theorem

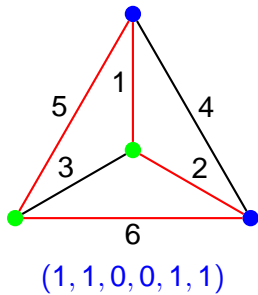
$|T(G; -1, -1)| = 2^{k-1}$ where k is the number of components of the knot.

Bicycles

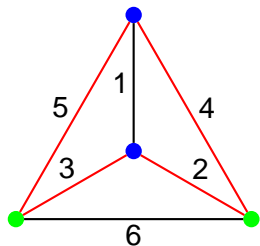


A **bicycle** is a subset A of edges such that every vertex is incident with an even number of edges from A and we can two-colour the vertices of G so that adjacent vertices receive the same colour if and only if they are joined by an edge from A .

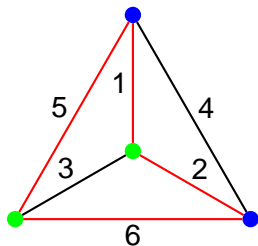
Bicycles



Bicycles

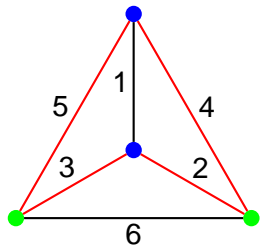


$(0, 1, 1, 1, 1, 0)$

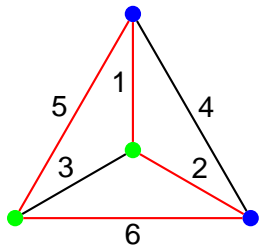


$(1, 1, 0, 0, 1, 1)$

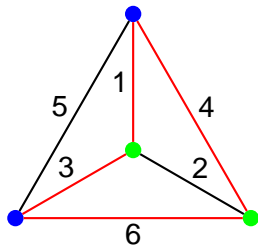
Bicycles



$(0, 1, 1, 1, 1, 0)$

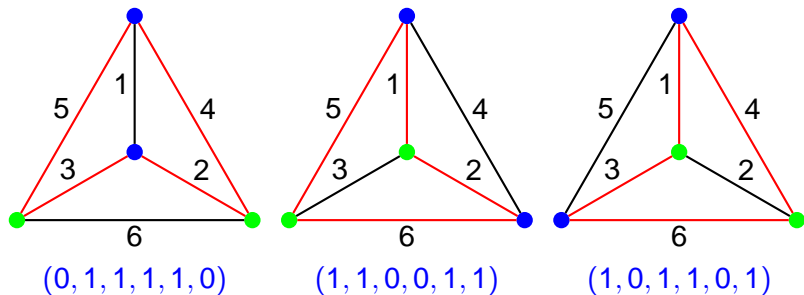


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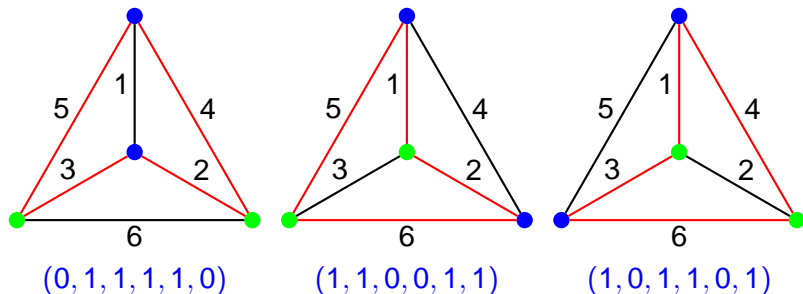
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Bicycles



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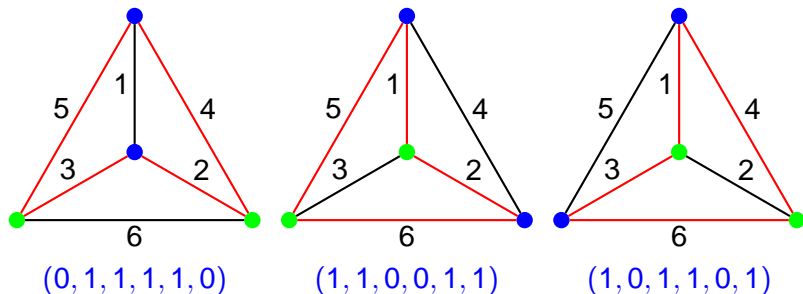
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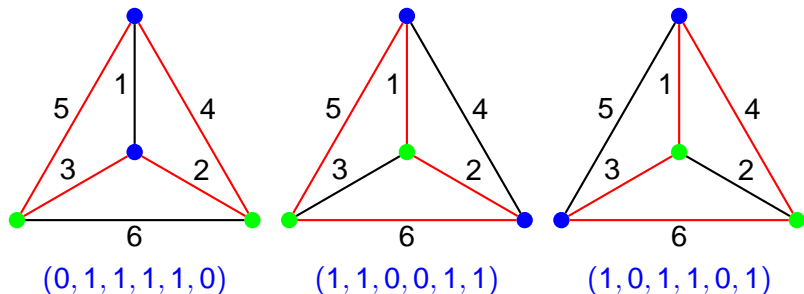


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We will say that G is **bicycle-covered** if every edge lies in a bicycle.

A Conjecture

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$$t(K_3; z) = -1 + (z + 1)^2$$

$$t(K_4; z) = 4 - 10(z + 1) + \dots$$

$$t(K_5; z) = 1 + \dots$$

$$t(K_6; z) = -16 + 80(z + 1) - 110(z + 1)^2 + 45(z + 1)^3 + \dots$$

$$t(K_7; z) = 1 + \dots$$

$$t(K_8; z) = 64 - 592(z + 1) + \dots .$$

A Conjecture

Let $t(G; z) = T(G; z, z)$.

Open Question

For which class of graphs is it true that if $t(G; z) = \sum_j a_j(z+1)^j$, then $2^{b(G)-j} | a_j$ for $j = 1, \dots, k$?

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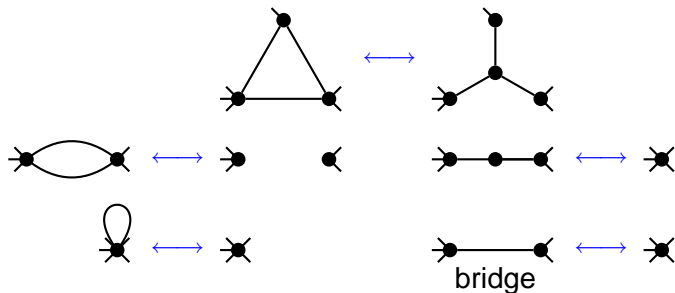
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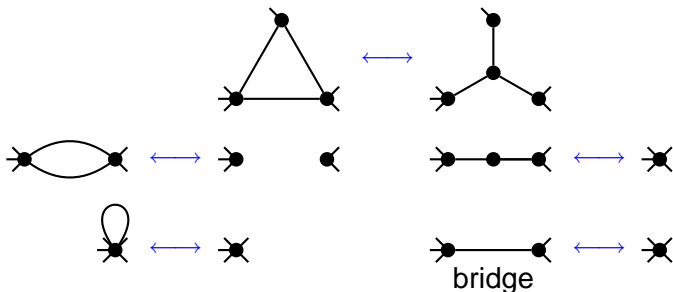
Knot graphs

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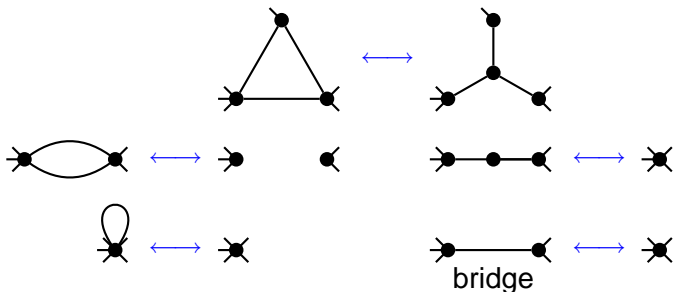


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K_6 is not a knot graph but every planar graph is a knot graph.

What can we show?

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- Deleting / contracting an edge that does not lie in a bicycle cannot decrease $b(G)$.
- Use induction and $t(G) = t(G - e) + t(G/e)$ to see that $t'(G; -1)$ is divisible by $2^{b(G)-1}$ in any planar graph.

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- Is the conjecture true for planar graphs?
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- Just one of the many open problems concerning $T \dots$