The Tutte Polynomial

Steve Noble

27/11/2010

Research at Brunel

- Brunel is in Uxbridge, as far west in London as it is possible to go!
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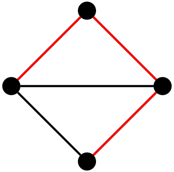
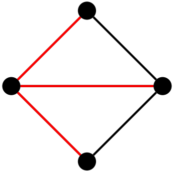
- Please see our brochure or our webpage at `www.brunel.ac.uk/siscm/mathematical-sciences.html`

Counting Spanning Trees



$\tau(G)$

Counting Spanning Trees



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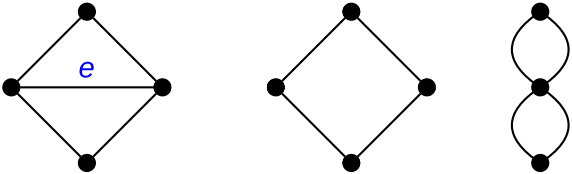

$\tau(G)$ $\tau(G - e)$ $\tau(G/e)$

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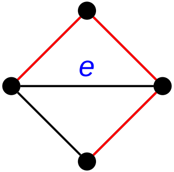

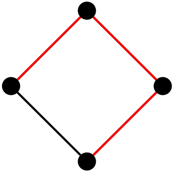

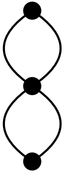

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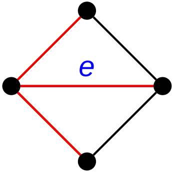

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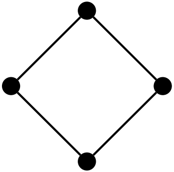

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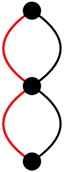

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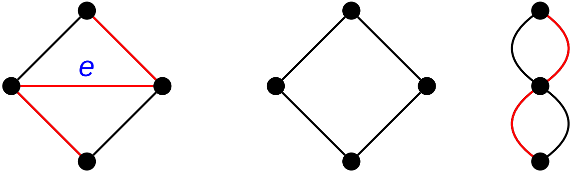

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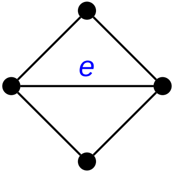
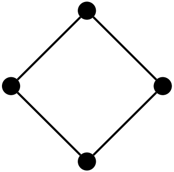
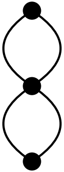
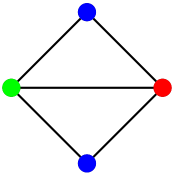






$$\tau(G) \quad = \quad \tau(G - e) \quad + \quad \tau(G/e)$$

Chromatic Polynomial



$\chi(G;\lambda)$

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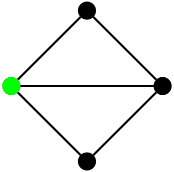
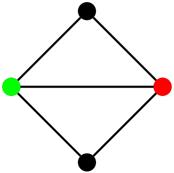


$\chi(G;\lambda)$

$$\chi(G;\lambda) = \lambda \qquad\qquad .$$

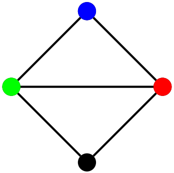
Chromatic Polynomial



$\chi(G; \lambda)$

$$\chi(G; \lambda) = \lambda(\lambda - 1) \qquad .$$

Chromatic Polynomial



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$$\chi(G;\lambda) = \lambda(\lambda-1)(\lambda-2)\ .$$

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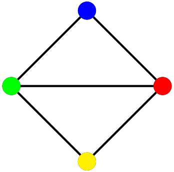

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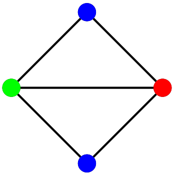

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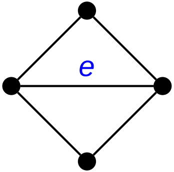
$$\chi(G;\lambda) = \lambda(\lambda - 1)(\lambda - 2)^2.$$

Chromatic Polynomial



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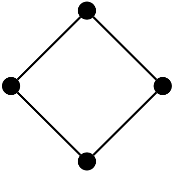

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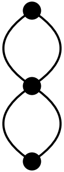

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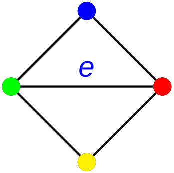

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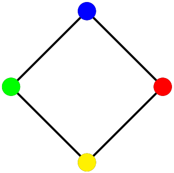

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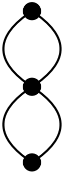

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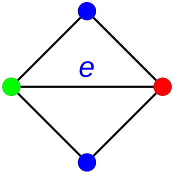

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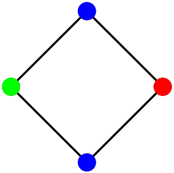

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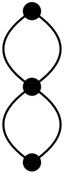

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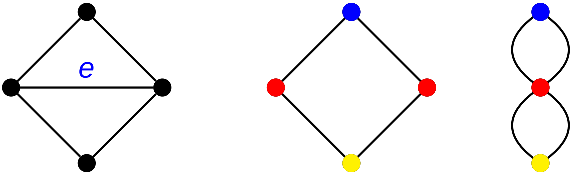

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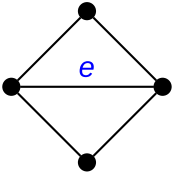

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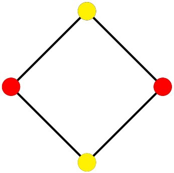

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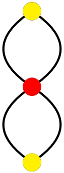

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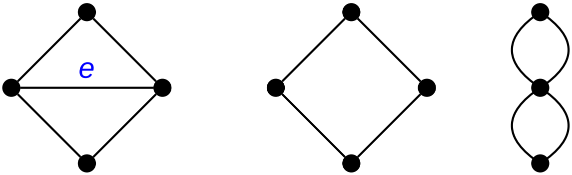

$$\chi(G;\lambda) \quad = \quad \chi(G-e;\lambda) \; - \; \chi(G/e;\lambda)$$

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Definition (Tutte, 1948)

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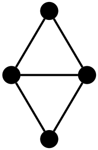
- If G is a one edge graph and the edge is a bridge then $T(G;x,y) = x$.
- If e is an edge of G that is neither a loop nor a bridge then

$$T(G;x,y) = T(G-e;x,y) + T(G/e;x,y).$$

An Example



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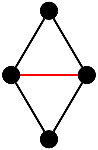

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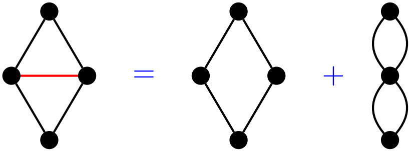

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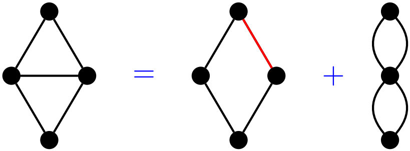

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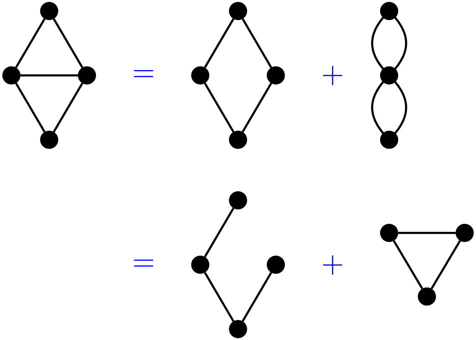

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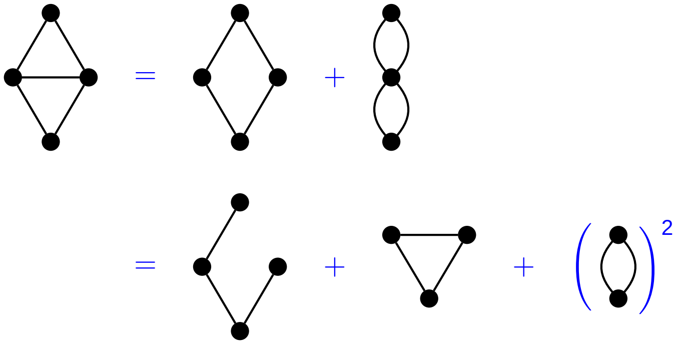

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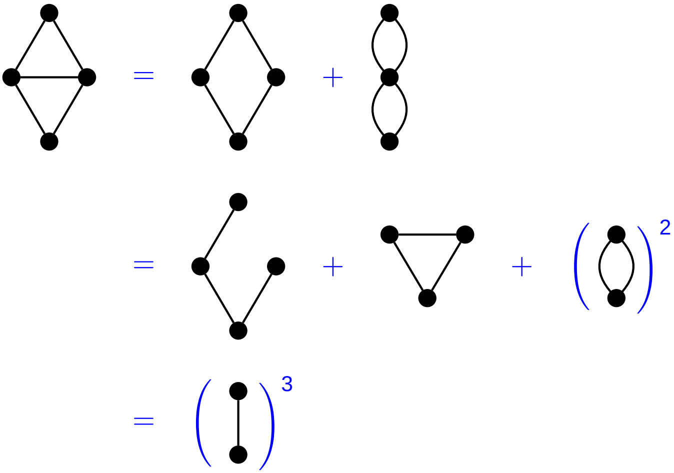

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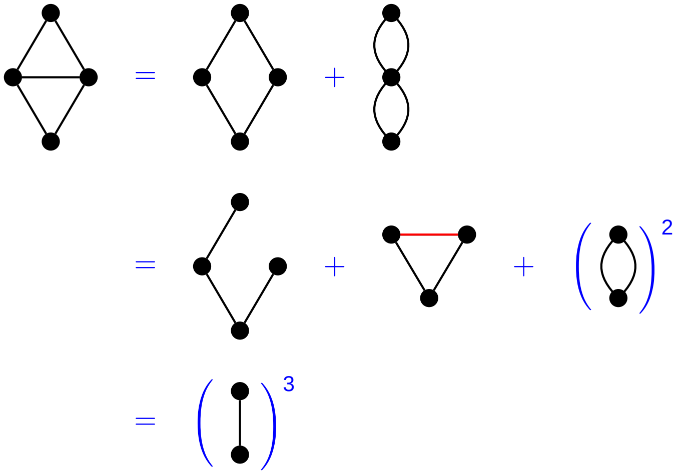

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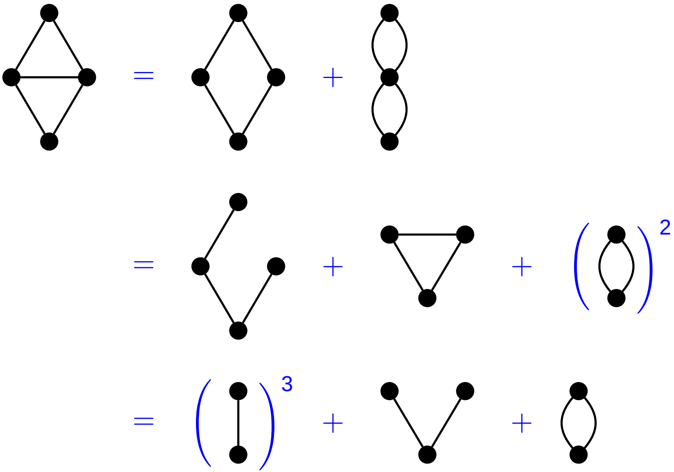

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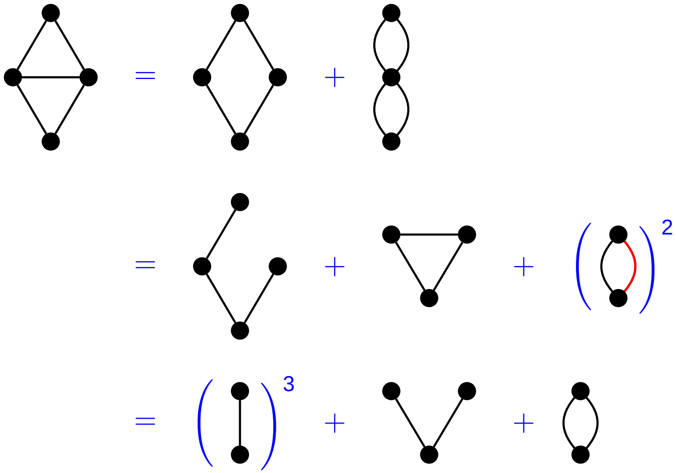

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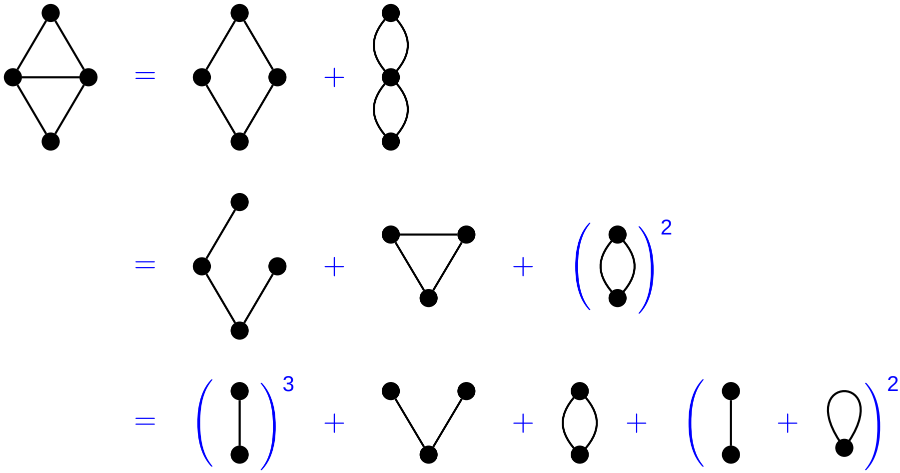

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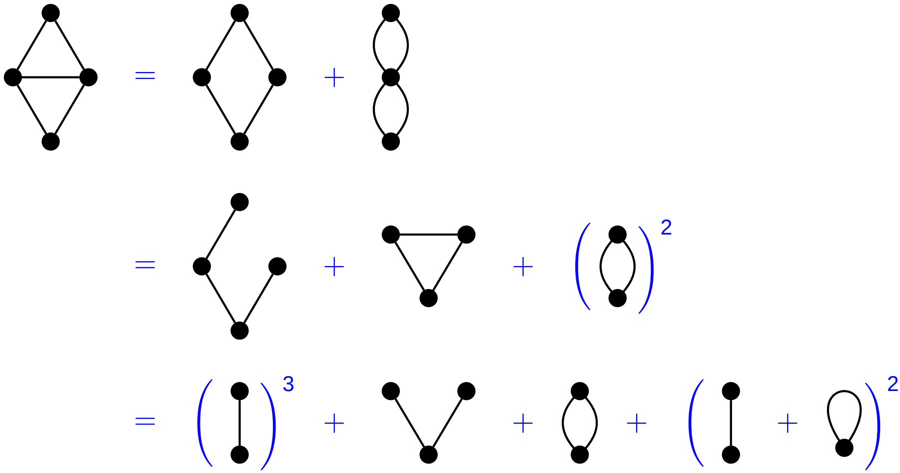
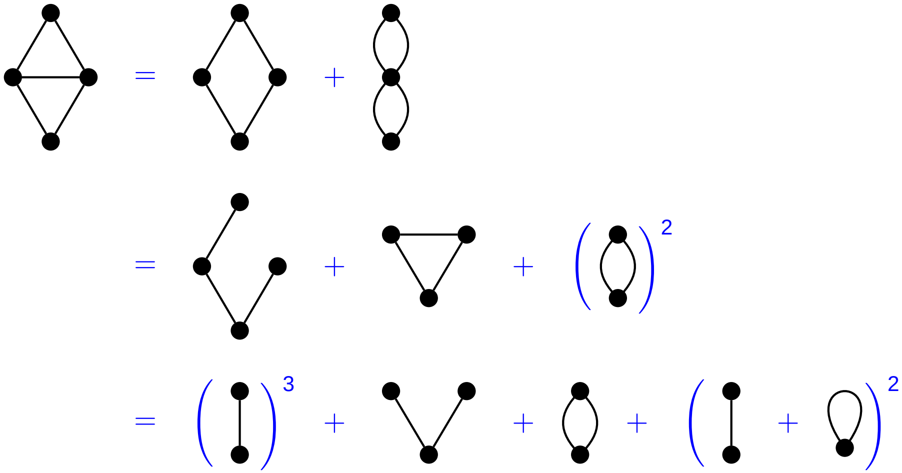
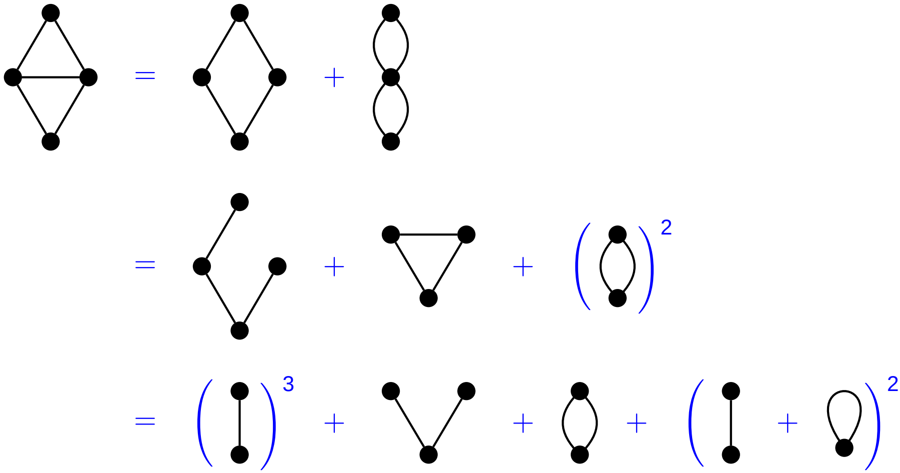


$$= x^3$$



$$= x^3 + x^2$$

An Example



$$= x^3 + x^2 + (x+y) + (x+y)^2$$

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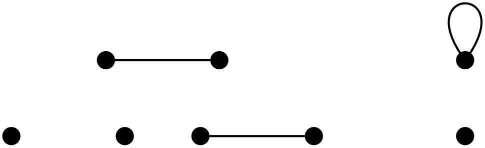

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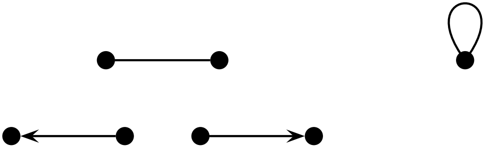

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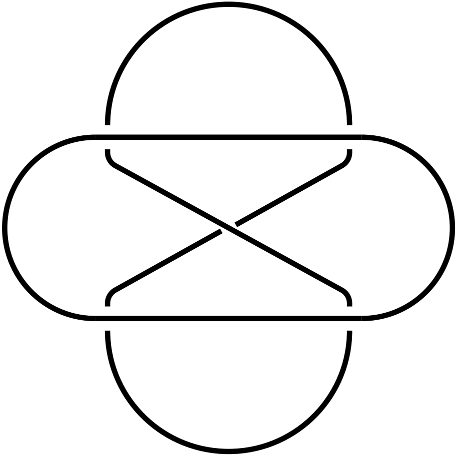
- $T(G;2,1)$ is the number of forests and the number of score vectors of G.
- $T(G;2,0)$ is the number of acyclic orientations of G.
- Other specializations to applications from operational research, statistical physics and topology.

Knots



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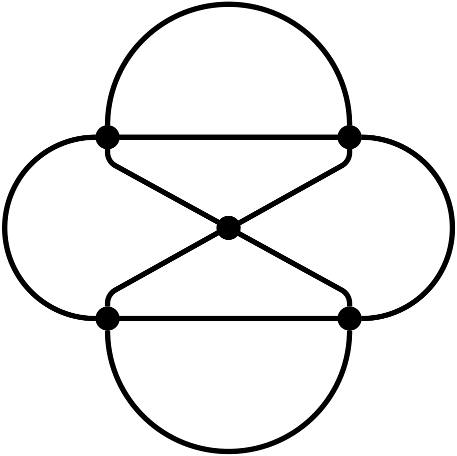

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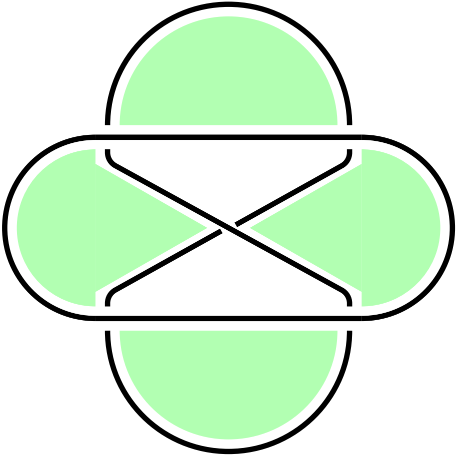

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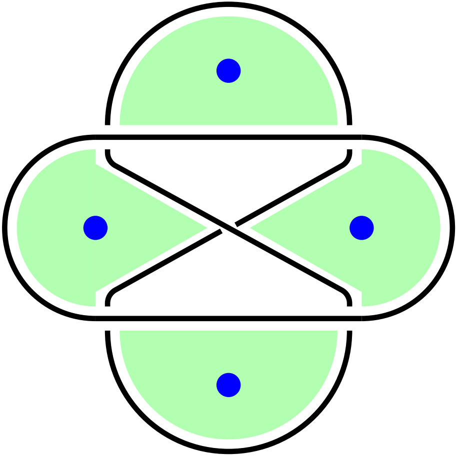

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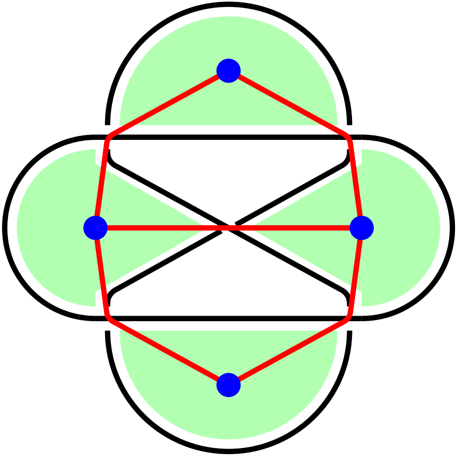

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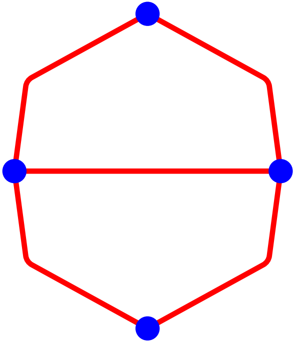

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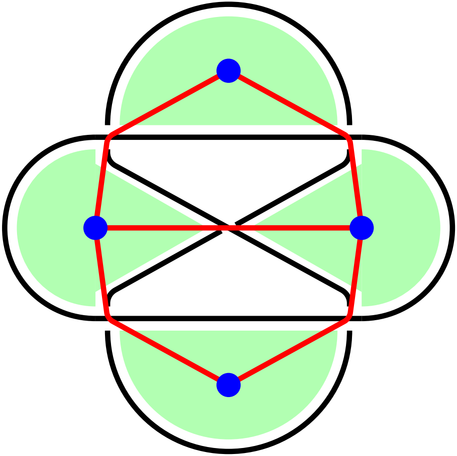
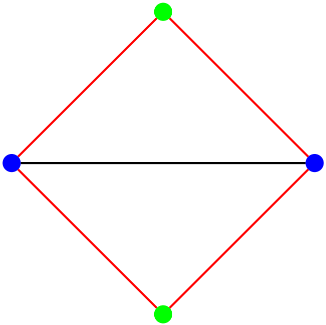


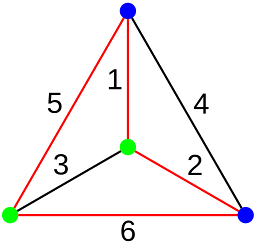
Theorem

$|T(G;-1,-1)| = 2^{k-1}$ *where* k *is the number of components of the knot.*



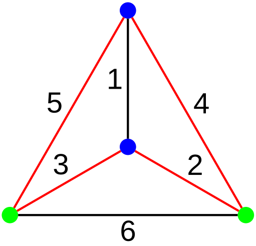
A bicycle is a subset A of edges such that every vertex is incident with an even number of edges from A and we can two-colour the vertices of G so that adjacent vertices receive the same colour if and only if they are joined by an edge from A.

Bicycles

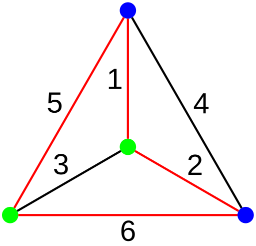


$(1, 1, 0, 0, 1, 1)$

Bicycles



$(0, 1, 1, 1, 1, 0)$



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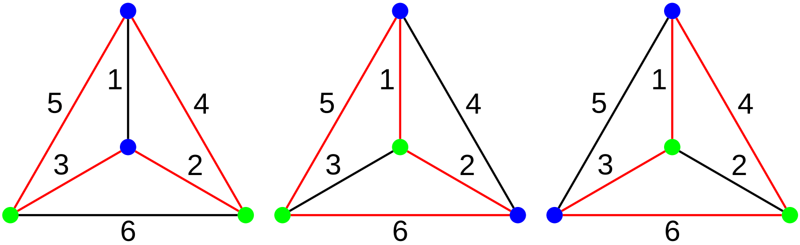

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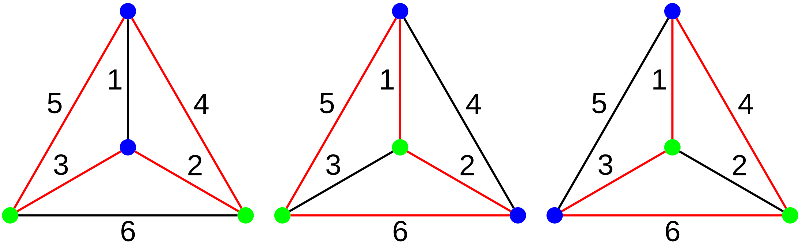

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The incidence vectors of bicycles form a vector space over $GF(2)$.

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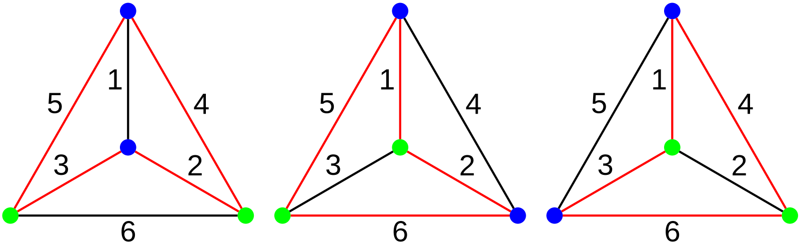

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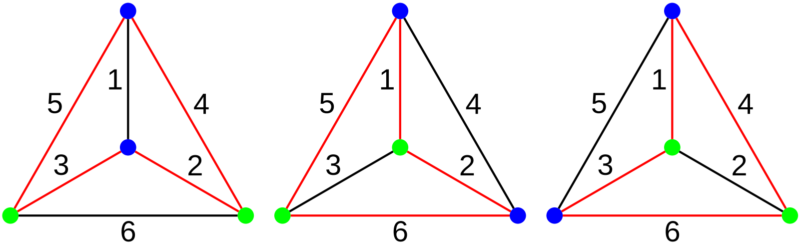

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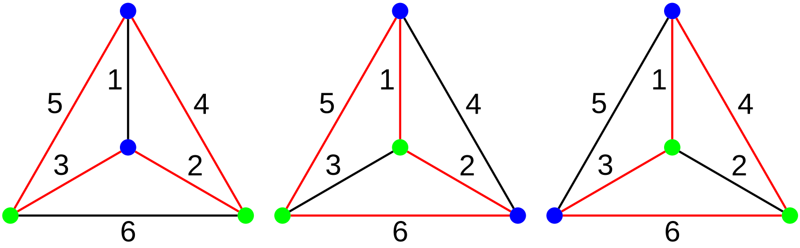

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We will say that G is bicycle-covered if every edge lies in a bicycle.

A Conjecture

Let $t(G; z) = T(G; z, z)$.

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Conjecture (Las Vergnas (1988))

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$$t(K_3;z) = -1 + (z+1)^2$$
$$t(K_4;z) = 4 - 10(z+1) + \cdots$$
$$t(K_5;z) = 1 + \cdots$$
$$t(K_6;z) = -16 + 80(z+1) - 110(z+1)^2 + 45(z+1)^3 + \cdots$$
$$t(K_7;z) = 1 + \cdots$$
$$t(K_8;z) = 64 - 592(z+1) + \cdots .$$

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Open Question

For which class of graphs is it true that if $t(G;z) = \sum_j a_j(z+1)^j$, then $2^{b(G)-j} | a_j$ for $j = 1, \ldots, k$?

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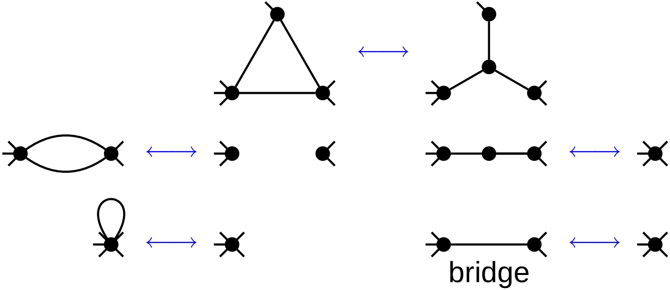
Knot graphs

A graph is a knot graph if it can be obtained from a graph with no edges using the following five local moves.



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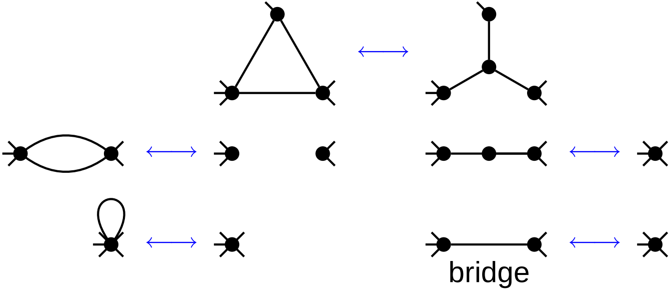

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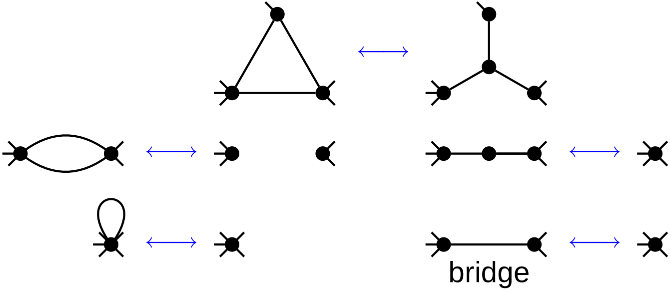

Theorem

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Theorem

K_6 is not a knot graph but every planar graph is a knot graph.

What can we show?

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- $t'(G;-1)$ is divisible by $2^{b(G)-1}$ in a bicycle-covered planar graph.
- Deleting / contracting an edge that does not lie in a bicycle cannot decrease $b(G)$.
- Use induction and $t(G) = t(G-e) + t(G/e)$ to see that $t'(G;-1)$ is divisible by $2^{b(G)-1}$ in any planar graph.

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- Just one of the many open problems concerning T ...