Quantum Mechanics

Consider the quantum harmonic oscillator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

Goal: Find E and $\psi(x)$ such that

$$H\psi(x) = E\psi(x) \quad , \quad \int_0^\infty |\psi(x)|^2 dx < \infty$$

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Wavefunctions

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{-1/4} H_n\left(\sqrt{\frac{m\omega}{2\hbar}}x\right) e^{-m\omega x^2/2\hbar}$$

and allowed energy levels

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, \ldots$$

Anharmonic oscillators

Consider the anharmonic oscillators

$$H = -\frac{d^2}{dx^2} + x^{2M}$$

Goal: Find E and $\psi(x)$ such that $\int_0^\infty |\psi(x)|^2 dx < \infty$

Numerical solution: all eigenvalues E_n are real for $M \geq 1$

Reality follows from Hermiticity of $H = H^\dagger$

Hermiticity $\Rightarrow$ real spectrum

SE $H\psi = E\psi$

$$\int \psi^* H\psi \, dx = \int \psi^* E\psi \, dx$$

Take complex conjugate

$$\int \psi^* H^\dagger \psi \, dx = \int \psi^* E^* \psi \, dx$$

Using $H^\dagger = H$

$$(E - E^*) \int \psi^* \psi \, dx = 0 \quad \Rightarrow \quad E = E^*$$

Non-hermitian anharmonic oscillators

Consider the complex anharmonic oscillators

$$H = -\frac{d^2}{dx^2} - (ix)^{2M}$$

Goal: Find E and $\psi(x)$ such that $\int_0^\infty |\psi(x)|^2 dx < \infty$

Non-hermitian anharmonic oscillators

Consider the complex anharmonic oscillators

$$H = -\frac{d^2}{dx^2} - (ix)^{2M}$$

Goal: Find E and $\psi(x)$ such that $\int_0^\infty |\psi(x)|^2 dx < \infty$

H not Hermitian: $H^\dagger \neq H$

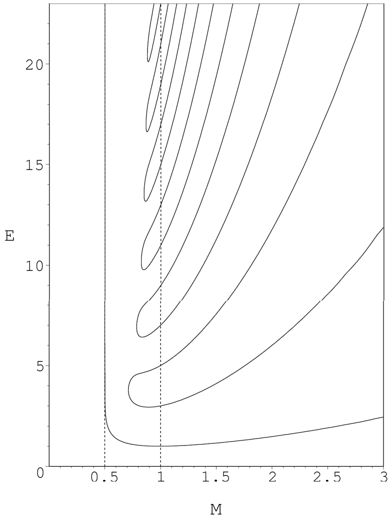
H is PT-symmetric: $P : x \to -x \quad , T : i \to -i$

PT-symmetry $\Rightarrow 0 = (E - E^*) \int \psi(-x)^* \psi(x) \, dx = 0$

A surprise

Numerics: eigenvalues all real for $M > 1$

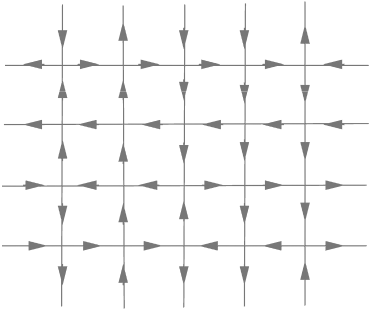
C.M. Bender and S. Boettcher Physics Review Letters 80 (1998) 5243



Proof of reality via 2D ice-like model (Dorey, Dunning, Tateo)

Oxygen atom = vertex
Bond with Hydrogen = link

Ice rule: oxygen atom has two close hydrogen and two far away



6-vertex model

Free energy

$$f =\sim -\frac{1}{N}\log t$$

in terms of transfer matrix

$$t(\nu) = a^N(\nu,\eta)\prod_{j=1}^{n} g(\nu_j - \nu) + b^N(\nu,\eta)\prod_{j=1}^{n} g(\nu - \nu_j) \ ,$$

IM/ODE correspondence

Six vertex model as $N \to \infty$

$$t(\nu)q(\nu) = e^{i\phi}q(\omega^2\nu) + e^{-i\phi}q(\omega^{-2}\nu)$$

Bethe roots satisfy

$$\prod_{n=1}^{\infty}\left(\frac{\nu_n - q^2\nu_k}{\nu_n - q^{-2}\nu_k}\right) = -e^{4\pi ip} \qquad ik1, \ldots$$

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Schrödinger equation

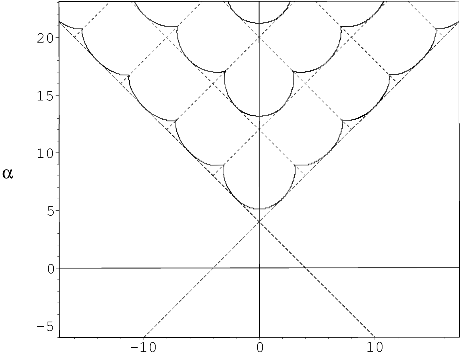
$$C(E)D(E) = \omega^{-1/2}D(\omega^{-2}E) + \omega^{1/2}D(\omega^2 E)$$

Eigenvalues of Bender-Boettcher problem satisfy $C(E_k) = 0$

$$\prod_{n=1}^{\infty}\left(\frac{\mathcal{E}_n - \omega^2 E_k}{\mathcal{E}_n - \omega^{-2} E_k}\right) = -\omega^{-1} \qquad k = 1, \ldots$$

Generalised problem (P. Dorey, C. Dunning and R. Tateo, J. Phys. A34 (2001) 5697)

$$H = \frac{d^2}{dx^2} - (ix)^{2M} - \alpha(ix)^{M-1} + \frac{\lambda^2 - 1/4}{x^2}$$



- *real* for $M > 1$ and $\alpha < M + 1 + |\lambda|$;
- *positive* for $M > 1$ and $\alpha < M + 1 - |\lambda|$.