LTCC: The Invariant Theory of Finite Groups

Prof. Peter Fleischmann and Dr Jim Shank, University of Kent

Autumn 2009
Week 1, Hour 2

Introduction

The transfer (or trace) is an $\mathbb{F}[V]^G$-module homomorphism

$$\begin{aligned} \mathrm{tr} : \mathbb{F}[V] &\to \mathbb{F}[V]^G \\ f &\mapsto \sum_{\tau \in G} (f)\tau. \end{aligned}$$

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which gives an $\mathbb{F}[V]^G$-module splitting

$$\mathbb{F}[V] = \mathbb{F}[V]^G \oplus \ker(\mathcal{R}).$$

For a homogeneous system of parameters $\{h_1 \ldots, h_n\} \subset \mathbb{F}[V]^G$, define $A := \mathbb{F}[h_1 \ldots, h_n]$ and $J := (h_1, \ldots, h_n)\mathbb{F}[V]$.

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Computing a Gröbner basis for J gives a basis $\mathcal{B}$ for $\mathbb{F}[V]$ as a free A-module: take $\mathcal{B}$ to be the set of monomials not divisible by the lead monomial of an element from J. Then $\mathbb{F}[V]^G$ is generated by

$$\{h_1, \ldots, h_n\} \cup \{\mathcal{R}(\alpha) \mid \alpha \in \mathcal{B}\}.$$

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In the modular case, $\mathbb{F}[V]^G$ is often not Cohen-Macaulay and, while algorithms for constructing generating sets are known (Kemper 1996), the calculations can be slow and memory intensive.

Links with Algebraic Topology and Group Cohomology

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What can we say about $H^*(X, \mathbb{F}_p)^G$?

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$$X := \{(y, e) \in Y \times EG \mid f(y) = \pi(e)\}.$$

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The *classifying space* BG is the orbit space EG/G.

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Dwyer-Miller-Wilkerson: For odd p, X suitably nice with $H^*(X, \mathbb{F}_p)$ a noetherian normal integral domain of transcendence degree n over $\mathbb{F}_p$, then for some $H \leq \mathrm{GL}_n(\mathbb{F}_p)$,

$$H^*(X, \mathbb{F}_p) \cong H^*(BT_n, \mathbb{F}_p)^H.$$

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In the modular case, we have a map

$$H^*(BG, \mathbb{F}_p) \to H^*(BT, \mathbb{F}_p)^W \subset H^*(BT, \mathbb{F}_p).$$

For finite G,

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Then $W(H)$ acts on $H^*(H, \mathbb{F}_p)$ and the restriction, the map induced by inclusion, factors:

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Let P denote a Sylow p-subgroup of G. If P is abelian then

$$H^*(G, \mathbb{F}_p) \cong H^*(P, \mathbb{F}_p)^{W(P)}.$$

Example: $G = \mathbb{Z}/2$ acts by the antipodal map on S^n, giving

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For a rank r elementary abelian 2-group we have $B(\mathbb{Z}/2)^r = (\mathbb{R}P^\infty)^r$ and

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Analogous constructions for odd primes gives

$$H^*((\mathbb{Z}/p)^r, \mathbb{F}_p) = \Lambda_p(x_1, \ldots, x_r) \otimes \mathbb{F}_p[y_1, \ldots, y_r]$$

with $y_i = \beta(x_i)$, where β is the Bockstein operation.

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Compute the relation.

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ϕ is injective and the image of ϕ lies in the kernel of ρ.

Example: $H^*(\Sigma_4, \mathbb{F}_2)$ (Adem & Milgram Ch. VI)

Denote $E_1 =< (12), (34) >$, $E_2 =< (12)(34), (14)(23) >$ and $E = E_1 \cap E_2 =< (12)(34) >$.

Then $N_1 := N(E_1)/E_1 \cong \mathbb{Z}/2$ and $N_2 := N(E_2)/E_2 \cong \Sigma_3 \cong \mathrm{GL}_2(\mathbb{F}_2)$.

Restriction gives

$$H^*(\Sigma_4, \mathbb{F}_2) \xrightarrow{\phi} H^*(E_1, \mathbb{F}_2)^{N_1} \oplus H^*(E_2, \mathbb{F}_2)^{N_2}$$

and

$$H^*(E_1, \mathbb{F}_2) \oplus H^*(E_2, \mathbb{F}_2) \xrightarrow{\rho} H^*(E, \mathbb{F}_2) = \mathbb{F}_2[z].$$

ϕ is injective and the image of ϕ lies in the kernel of ρ.

We need to compute the modular invariant rings

$$H^*(E_1)^{N_1} = \mathbb{F}_2[x_1, y_1]^{\mathbb{Z}/2} \text{ and } H^*(E_2)^{N_2} = \mathbb{F}_2[x_2, y_2]^{\mathrm{GL}_2(\mathbb{F}_2)}.$$

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A little calculation gives $\rho(x_1) = z$, $\rho(y_1) = z$, $\rho(x_2) = z$ and $\rho(y_2) = 0$.

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Thus $\rho(e_1) = 0$, $\rho(e_2) = z^2$, $\rho(d_2) = z^2$, and $\rho(d_3) = 0$.

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Thus $\rho(e_1) = 0$, $\rho(e_2) = z^2$, $\rho(d_2) = z^2$, and $\rho(d_3) = 0$.

$H^*(\Sigma_4, \mathbb{F})$ is generated by elements f_1, f_2 and f_3 with $\phi(f_1) = e_1$, $\phi(f_2) = e_2 + d_2$, and $\phi(f_3) = d_3$, subject to the relation $f_1f_3 = 0$.