Invariant Theory of Finite Groups

Peter Fleischmann / Jim Shank

School of Mathematics, Statistics and Actuarial Science

University of Kent

Week 5

London, 2 November 2009

Recall from Week 3

V finite dimensional $\mathbb{F}$ - vector space, $G \leq \mathrm{GL}(V)$ finite;
$A := \mathbb{F}[V] = \mathrm{Sym}(V^*) \cong \mathbb{F}[x_1, \cdots, x_n]$, polynomial ring;

$$A^G := \{a \in A \mid g(a) = a,\ \forall g \in G\}, (\mathbb{N}_0 - \mathrm{graded})$$

Theorem (Chevalley-Shephard-Todd):
Consider the following statements:

1. ${}_{A^G}A$ is free.
2. A^G is a polynomial ring.
3. $G \leq \mathrm{GL}(V)$ is generated by pseudo-reflections.

Then the following implications hold: 1. $\iff$ 2. $\Rightarrow$ 3.
If $|G| \in \mathbb{F}^*$ then: 1. $\iff$ 2. $\iff$ 3.

Modular Counterexample

Let $\mathbb{F} := \mathbb{F}_p[\epsilon]$ with $\epsilon^p - \epsilon = 1$ and define $G := \langle r_1, r_2, r_3 \rangle$ with

$$r_1 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, r_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$r_3 := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \epsilon \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Then $\mathbb{F}[\mathbb{F}^4]^G$ is not a polynomial ring.

(Example due to H.E.A. Campbell and D. Wehlau)

Exercises

1 Let R be a normal integral domain and $G \leq \mathrm{Aut}(R)$. Then R^G is also a normal integral domain.

2 Let $A := \mathbb{F}[V]$, $\mathbb{L} := \mathrm{Quot}(A)$, $G \leq \mathrm{GL}(V)$ finite. Show that $\mathbb{L} \geq \mathbb{L}^G = \mathrm{Quot}(A^G) =: \mathbb{K}$ is galois with group G. Conclude that for $H \leq G$, $H = G \iff A^H = A^G$.

3 Show that the following are equivalent:

(a) A^G is a polynomial ring.

(b) There is an hsop $\{h_1, \cdots, h_n\}$ of A^G with $|G| = \prod_{i=1}^n \deg h_i$.

In particular $A^G = \mathbb{F}[h_1, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n \deg h_i$.

Exercises

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:
(a) $W \in \mathbb{F}G - \mathtt{mod}$ of dimension $|G|$.
(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

(Hint: Use normal basis theorem from Galois theory, Maschke's theorem from representation theory and use the fact that for finite dimensional $X, Y \in \mathbb{F}G - \mathtt{mod}$ and any field extension $\mathbb{S} \geq \mathbb{F}$

$$\mathbb{S} \otimes_{\mathbb{F}} X \cong \mathbb{S} \otimes_{\mathbb{F}} Y$$

as $\mathbb{S}G$-modules $\iff X \cong Y$ as $\mathbb{F}G$-modules.)

Exercise 1

1 Let R be a normal integral domain and $G \leq \mathrm{Aut}(R)$. Then R^G is also a normal integral domain.

Exercise 1

1 Let R be a normal integral domain and $G \leq \mathrm{Aut}(R)$. Then R^G is also a normal integral domain.

Solution : Show that R^G is integrally closed in $\mathrm{Quot}(R^G)$:

Exercise 1

1 Let R be a normal integral domain and $G \leq \mathrm{Aut}(R)$. Then R^G is also a normal integral domain.

Solution : Show that R^G is integrally closed in $\mathrm{Quot}(R^G)$:

Let $x = f/g \in \mathrm{Quot}(R^G) \leq \mathrm{Quot}(R)$ integral $/R^G$; $\Rightarrow$

Exercise 1

1 Let R be a normal integral domain and $G \leq \mathrm{Aut}(R)$. Then R^G is also a normal integral domain.

Solution : Show that R^G is integrally closed in $\mathrm{Quot}(R^G)$:

Let $x = f/g \in \mathrm{Quot}(R^G) \leq \mathrm{Quot}(R)$ integral $/R^G$; $\Rightarrow$

x integral $/R \Rightarrow x \in R$ (as R normal).

Exercise 1

1 Let R be a normal integral domain and $G \leq \mathrm{Aut}(R)$. Then R^G is also a normal integral domain.

Solution : Show that R^G is integrally closed in $\mathrm{Quot}(R^G)$:

Let $x = f/g \in \mathrm{Quot}(R^G) \leq \mathrm{Quot}(R)$ integral $/R^G$; $\Rightarrow$

x integral $/R \Rightarrow x \in R$ (as R normal).

$\Rightarrow x \in R \cap \mathrm{Quot}(R^G) = R^G$.

Exercise 2

2 Let $A := \mathbb{F}[V]$, $\mathbb{L} := \mathrm{Quot}(A)$, $G \leq \mathrm{GL}(V)$ finite. Show that $\mathbb{L} \geq \mathbb{L}^G = \mathrm{Quot}(A^G) =: \mathbb{K}$ is galois with group G. Conclude that for $H \leq G$, $H = G \iff A^H = A^G$.

Exercise 2

2 Let $A := \mathbb{F}[V]$, $\mathbb{L} := \mathrm{Quot}(A)$, $G \leq \mathrm{GL}(V)$ finite. Show that $\mathbb{L} \geq \mathbb{L}^G = \mathrm{Quot}(A^G) =: \mathbb{K}$ is galois with group G. Conclude that for $H \leq G$, $H = G \iff A^H = A^G$.

Solution : Let $f = p/q \in \mathbb{L}^G$ with $p, q \in A$.

Exercise 2

2 Let $A := \mathbb{F}[V]$, $\mathbb{L} := \mathrm{Quot}(A)$, $G \leq \mathrm{GL}(V)$ finite. Show that $\mathbb{L} \geq \mathbb{L}^G = \mathrm{Quot}(A^G) =: \mathbb{K}$ is galois with group G. Conclude that for $H \leq G$, $H = G \iff A^H = A^G$.

Solution : Let $f = p/q \in \mathbb{L}^G$ with $p, q \in A$.

$f = \frac{p \cdot \prod_{1 \neq g \in G} {}^g q}{\prod_{g \in G} {}^g q} = p'/q'$ with $q' \in A^G$, $p' \in A$.

Exercise 2

2 Let $A := \mathbb{F}[V]$, $\mathbb{L} := \mathrm{Quot}(A)$, $G \leq \mathrm{GL}(V)$ finite. Show that $\mathbb{L} \geq \mathbb{L}^G = \mathrm{Quot}(A^G) =: \mathbb{K}$ is galois with group G. Conclude that for $H \leq G$, $H = G \iff A^H = A^G$.

Solution : Let $f = p/q \in \mathbb{L}^G$ with $p, q \in A$.

$f = \frac{p \cdot \prod_{1 \neq g \in G} {}^g q}{\prod_{g \in G} {}^g q} = p'/q'$ with $q' \in A^G$, $p' \in A$.

Hence $p' = q' f \in A \cap \mathbb{L}^G = A^G$ and $f \in \mathrm{Quot}(A^G)$.

Exercise 2

2 Let $A := \mathbb{F}[V]$, $\mathbb{L} := \mathrm{Quot}(A)$, $G \leq \mathrm{GL}(V)$ finite. Show that $\mathbb{L} \geq \mathbb{L}^G = \mathrm{Quot}(A^G) =: \mathbb{K}$ is galois with group G. Conclude that for $H \leq G$, $H = G \iff A^H = A^G$.

Solution : Let $f = p/q \in \mathbb{L}^G$ with $p, q \in A$.

$f = \frac{p \cdot \prod_{1 \neq g \in G} {}^g q}{\prod_{g \in G} {}^g q} = p'/q'$ with $q' \in A^G$, $p' \in A$.

Hence $p' = q' f \in A \cap \mathbb{L}^G = A^G$ and $f \in \mathrm{Quot}(A^G)$.

Clearly $\mathrm{Quot}(A^G) \leq \mathbb{L}^G$, so $\mathrm{Quot}(A^G) = \mathbb{L}^G$.

Exercise 2

2 Let $A := \mathbb{F}[V]$, $\mathbb{L} := \mathrm{Quot}(A)$, $G \leq \mathrm{GL}(V)$ finite. Show that $\mathbb{L} \geq \mathbb{L}^G = \mathrm{Quot}(A^G) =: \mathbb{K}$ is galois with group G. Conclude that for $H \leq G$, $H = G \iff A^H = A^G$.

Solution : Let $f = p/q \in \mathbb{L}^G$ with $p, q \in A$.

$f = \frac{p \cdot \prod_{1 \neq g \in G} {}^g q}{\prod_{g \in G} {}^g q} = p'/q'$ with $q' \in A^G$, $p' \in A$.

Hence $p' = q' f \in A \cap \mathbb{L}^G = A^G$ and $f \in \mathrm{Quot}(A^G)$.

Clearly $\mathrm{Quot}(A^G) \leq \mathbb{L}^G$, so $\mathrm{Quot}(A^G) = \mathbb{L}^G$.

Rest by standard Galois theory (Artin's theorem, Galois correspondence).

Exercise 3

3 Show that the following are equivalent:

(a): A^G is a polynomial ring.

(b): There is an hsop $\{h_1, \cdots, h_n\}$ of A^G with $|G| = \prod_{i=1}^n \deg h_i$.

In particular $A^G = \mathbb{F}[h_1, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n \deg h_i$.

Exercise 3

3 Show that the following are equivalent:

(a): A^G is a polynomial ring.

(b): There is an hsop $\{h_1, \cdots, h_n\}$ of A^G with $|G| = \prod_{i=1}^n \deg h_i$.

In particular $A^G = \mathbb{F}[h_1, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n \deg h_i$.

Solution : Recall , if $\mathcal{P} = \mathbb{F}[h_1, \cdots, h_n] \leq A^G$ is hsop-algebra, then

Exercise 3

3 Show that the following are equivalent:

(a): A^G is a polynomial ring.

(b): There is an hsop $\{h_1, \cdots, h_n\}$ of A^G with $|G| = \prod_{i=1}^n \deg h_i$.

In particular $A^G = \mathbb{F}[h_1, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n \deg h_i$.

Solution : Recall , if $\mathcal{P} = \mathbb{F}[h_1, \cdots, h_n] \leq A^G$ is hsop-algebra, then

$\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})]$.

Exercise 3

3 Show that the following are equivalent:

(a): A^G is a polynomial ring.

(b): There is an hsop $\{h_1, \cdots, h_n\}$ of A^G with $|G| = \prod_{i=1}^n \deg h_i$.

In particular $A^G = \mathbb{F}[h_1, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n \deg h_i$.

Solution : Recall , if $\mathcal{P} = \mathbb{F}[h_1, \cdots, h_n] \leq A^G$ is hsop-algebra, then

$\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})]$.

(a) $\Rightarrow$ can choose $\mathcal{P} = A^G$, hence $\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathbb{L}^G] = |G|$.

Exercise 3

3 Show that the following are equivalent:

(a): A^G is a polynomial ring.

(b): There is an hsop $\{h_1, \cdots, h_n\}$ of A^G with $|G| = \prod_{i=1}^n \deg h_i$.

In particular $A^G = \mathbb{F}[h_1, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n \deg h_i$.

Solution : Recall , if $\mathcal{P} = \mathbb{F}[h_1, \cdots, h_n] \leq A^G$ is hsop-algebra, then

$\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})]$.

(a) $\Rightarrow$ can choose $\mathcal{P} = A^G$, hence $\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathbb{L}^G] = |G|$.

(b) $\Rightarrow$ $|G| = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})] = [\mathbb{L} : \mathbb{L}^G] \cdot [\mathbb{L}^G : \mathrm{Quot}(\mathcal{P})]$.

Exercise 3

3 Show that the following are equivalent:

(a): A^G is a polynomial ring.

(b): There is an hsop $\{h_1, \cdots, h_n\}$ of A^G with $|G| = \prod_{i=1}^n \deg h_i$.

In particular $A^G = \mathbb{F}[h_1, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n \deg h_i$.

Solution : Recall , if $\mathcal{P} = \mathbb{F}[h_1, \cdots, h_n] \leq A^G$ is hsop-algebra, then

$\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})]$.

(a) $\Rightarrow$ can choose $\mathcal{P} = A^G$, hence $\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathbb{L}^G] = |G|$.

(b) $\Rightarrow |G| = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})] = [\mathbb{L} : \mathbb{L}^G] \cdot [\mathbb{L}^G : \mathrm{Quot}(\mathcal{P})]$.

$\Rightarrow \mathbb{L}^G = \mathrm{Quot}(A^G) = \mathrm{Quot}(\mathcal{P})$.

Exercise 3

3 Show that the following are equivalent:

(a): A^G is a polynomial ring.

(b): There is an hsop $\{h_1, \cdots, h_n\}$ of A^G with $|G| = \prod_{i=1}^n \deg h_i$.

In particular $A^G = \mathbb{F}[h_1, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n \deg h_i$.

Solution : Recall , if $\mathcal{P} = \mathbb{F}[h_1, \cdots, h_n] \leq A^G$ is hsop-algebra, then

$\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})]$.

(a) $\Rightarrow$ can choose $\mathcal{P} = A^G$, hence $\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathbb{L}^G] = |G|$.

(b) $\Rightarrow |G| = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})] = [\mathbb{L} : \mathbb{L}^G] \cdot [\mathbb{L}^G : \mathrm{Quot}(\mathcal{P})]$.

$\Rightarrow \mathbb{L}^G = \mathrm{Quot}(A^G) = \mathrm{Quot}(\mathcal{P})$.

A^G integral $/\mathcal{P}$, $\mathcal{P}$ normal (=integrally closed in $\mathrm{Quot}(\mathcal{P})$),

Exercise 3

3 Show that the following are equivalent:

(a): A^G is a polynomial ring.

(b): There is an hsop $\{h_1, \cdots, h_n\}$ of A^G with $|G| = \prod_{i=1}^n \deg h_i$.

In particular $A^G = \mathbb{F}[h_1, \cdots, h_n]$ implies $|G| = \prod_{i=1}^n \deg h_i$.

Solution : Recall , if $\mathcal{P} = \mathbb{F}[h_1, \cdots, h_n] \leq A^G$ is hsop-algebra, then

$\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})]$.

(a) $\Rightarrow$ can choose $\mathcal{P} = A^G$, hence $\prod_{i=1}^n \deg h_i = [\mathbb{L} : \mathbb{L}^G] = |G|$.

(b) $\Rightarrow |G| = [\mathbb{L} : \mathrm{Quot}(\mathcal{P})] = [\mathbb{L} : \mathbb{L}^G] \cdot [\mathbb{L}^G : \mathrm{Quot}(\mathcal{P})]$.

$\Rightarrow \mathbb{L}^G = \mathrm{Quot}(A^G) = \mathrm{Quot}(\mathcal{P})$.

A^G integral $/\mathcal{P}$, $\mathcal{P}$ normal (=integrally closed in $\mathrm{Quot}(\mathcal{P})$),

$A^G = \mathcal{P}$.

Exercise 4

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:

(a) $W \in \mathbb{F}G - \mathtt{mod}$ of dimension $|G|$.

(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

Exercise 4

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:
(a) $W \in \mathbb{F}G - \texttt{mod}$ of dimension $|G|$.
(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

(a) Solution : $\mathrm{rank}({}_{A^G}A) = [\mathbb{L} : \mathbb{L}^G] = |G| =: n$.

Exercise 4

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:
(a) $W \in \mathbb{F}G - \texttt{mod}$ of dimension $|G|$.
(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

(a) Solution : $\mathrm{rank}({}_{A^G}A) = [\mathbb{L} : \mathbb{L}^G] = |G| =: n$.

If ${}_{A^G}A = \oplus_{i=1}^n A^G e_i$, then

$$W \cong \oplus_{i=1}^n A^G e_i / A^{G,+} e_i \cong \oplus_{i=1}^n \mathbb{F} e_i \in \mathbb{F}G - \texttt{mod}.$$

Exercise 4

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:

(a) $W \in \mathbb{F}G - \mathtt{mod}$ of dimension $|G|$.

(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

(b) Solution :

Exercise 4

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:
(a) $W \in \mathbb{F}G - \mathtt{mod}$ of dimension $|G|$.
(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

(b) Solution :

We have $1_{\mathbb{F}} \oplus X \cong W$. If $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, then $1_{\mathbb{F}}$ is projective in $\mathbb{F}G - \mathtt{mod}$, hence $|G| \in \mathbb{F}^*$.

Exercise 4

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:
(a) $W \in \mathbb{F}G - \mathtt{mod}$ of dimension $|G|$.
(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

(b) Solution :

We have $1_{\mathbb{F}} \oplus X \cong W$. If $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, then $1_{\mathbb{F}}$ is projective in $\mathbb{F}G - \mathtt{mod}$, hence $|G| \in \mathbb{F}^*$.

If $|G| \in \mathbb{F}^*$, then $A = A^{G,+}A \oplus X$ with $W \cong X \in \mathbb{F}G - \mathtt{mod}$.

Exercise 4

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:
(a) $W \in \mathbb{F}G - \texttt{mod}$ of dimension $|G|$.
(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

(b) Solution :

We have $1_{\mathbb{F}} \oplus X \cong W$. If $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, then $1_{\mathbb{F}}$ is projective in $\mathbb{F}G - \texttt{mod}$, hence $|G| \in \mathbb{F}^*$.

If $|G| \in \mathbb{F}^*$, then $A = A^{G,+}A \oplus X$ with $W \cong X \in \mathbb{F}G - \texttt{mod}$.

By Galois theory, there is a normal-basis generator $\ell \in \mathbb{L}$, hence

Exercise 4

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:
(a) $W \in \mathbb{F}G - \mathtt{mod}$ of dimension $|G|$.
(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

(b) Solution :

We have $1_\mathbb{F} \oplus X \cong W$. If $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, then $1_\mathbb{F}$ is projective in $\mathbb{F}G - \mathtt{mod}$, hence $|G| \in \mathbb{F}^*$.

If $|G| \in \mathbb{F}^*$, then $A = A^{G,+}A \oplus X$ with $W \cong X \in \mathbb{F}G - \mathtt{mod}$.

By Galois theory, there is a normal-basis generator $\ell \in \mathbb{L}$, hence for $Y := \oplus_{g \in G}\mathbb{F}\, {}^g\ell \subseteq \mathbb{L}$ we have $Y \cong {}_{\mathbb{F}G}\mathbb{F}G$ and

$$\mathbb{K} \otimes_\mathbb{F} Y \cong \mathbb{L} \cong \mathbb{K} \otimes_\mathbb{F} X \in \mathbb{K}G - \mathtt{mod};$$

Exercise 4

4 Assume that ${}_{A^G}A$ is free and let $W := A/A^{G,+}A$. Show:
(a) $W \in \mathbb{F}G-\texttt{mod}$ of dimension $|G|$.
(b) $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, the reg. rep. of G, if and only if $|G| \in \mathbb{F}^*$.

(b) Solution :

We have $1_{\mathbb{F}} \oplus X \cong W$. If $W \cong {}_{\mathbb{F}G}\mathbb{F}G$, then $1_{\mathbb{F}}$ is projective in $\mathbb{F}G-\texttt{mod}$, hence $|G| \in \mathbb{F}^*$.

If $|G| \in \mathbb{F}^*$, then $A = A^{G,+}A \oplus X$ with $W \cong X \in \mathbb{F}G-\texttt{mod}$.

By Galois theory, there is a normal-basis generator $\ell \in \mathbb{L}$, hence for $Y := \oplus_{g\in G}\mathbb{F}\,{}^g\ell \subseteq \mathbb{L}$ we have $Y \cong {}_{\mathbb{F}G}\mathbb{F}G$ and

$$\mathbb{K} \otimes_{\mathbb{F}} Y \cong \mathbb{L} \cong \mathbb{K} \otimes_{\mathbb{F}} X \in \mathbb{K}G-\texttt{mod};$$

$\Rightarrow Y \cong X \cong W$.

From Exercise 3:

To prove that A^G is polynomial ring:

1 Guess a set of n homogeneous algebra generators $a_1, \cdots, a_n \in A^G$.

2 Show that A is finite over $\mathbb{F}[a_1, \cdots, a_n]$.

3 Show that the degree of $\mathrm{Quot}(A)$ over $\mathbb{F}(a_1, \cdots, a_n)$ equals $|G|$, e.g. by showing that $\prod_{i=1}^{n} \deg a_i = |G|$.

Theorem (Kemper): sufficient to show 3. + $\{a_1, \cdots, a_n\}$ algebraically independent.

Examples

1. **Elementary symmetrics** $e_1, \cdots, e_n \in \mathbb{F}[X_1, \cdots, X_n]^{\Sigma_n}$,
clearly $\mathbb{F}[\underline{X}]$ integral over $\mathbb{F}[\underline{e}]$
$\prod_i \deg e_i = n! = |\Sigma_n| \Rightarrow \mathbb{F}[\underline{X}]^{\Sigma_n} = \mathbb{F}[\underline{e}]$
2. **Dickson - invariants** $d_{i,n} \in \mathbb{F}_q[V]^{\mathrm{GL}(V)}$ defined by
$F_n(T) := \prod_{v \in V^*}(T - v) = \sum_{i=0}^n d_{i,n} T^{q^{n-i}}$,
so $\mathbb{F}_q[V]$ integral over $\mathbb{F}_q[\underline{d}]$.
$\prod_i \deg d_{i,n} = |\mathrm{GL}(V)| \Rightarrow \mathbb{F}_q[V]^{\mathrm{GL}(V)} = \mathbb{F}_q[\underline{d}]$.
3. **Unipotent-invariants** $b_i \in \mathbb{F}_q[V]^{P_n(V)}$ with
$P_n(V) = \mathrm{Syl}_p(\mathrm{GL}(V))$, with orbit-products b_i of degree q^{i-1}.
As seen last week: $\{b_1, \cdots, b_n\}$ hsop with
$\prod_i \deg b_i = q^{\frac{n(n-1)}{2}} = |P_n(V)| \Rightarrow \mathbb{F}_q[V]^{P_n(V)} = \mathbb{F}_q[\underline{b}]$.

Nakajima groups

Let $\mathrm{char}\,\mathbb{F} = p > 0$, $P \leq \mathrm{GL}(V)$ a p-group. Can find basis $B = \{x_1, \cdots, x_n\}$ of V^* with $M_B(g)$ upper unitriangular $\forall g \in P$. Define

$$P_x := \mathrm{Stab}_P(x)$$

Nakajima groups

Let $\mathrm{char}\,\mathbb{F} = p > 0$, $P \leq \mathrm{GL}(V)$ a p-group. Can find basis $B = \{x_1, \cdots, x_n\}$ of V^* with $M_B(g)$ upper unitriangular $\forall g \in P$. Define

$$P_x := \mathrm{Stab}_P(x)$$

$$N(x) := N_{P_x}^{P}(x) = \text{orbit product of } x \in V^*,$$

Nakajima groups

Let $\mathrm{char}\,\mathbb{F} = p > 0$, $P \leq \mathrm{GL}(V)$ a p-group. Can find basis $B = \{x_1, \cdots, x_n\}$ of V^* with $M_B(g)$ upper unitriangular $\forall g \in P$. Define

$$P_x := \mathrm{Stab}_P(x)$$

$N(x) := N^P_{P_x}(x) =$ orbit product of $x \in V^*$,

- $P_i := \cap_{j \neq i} P_{x_j}$

Nakajima groups

Let $\operatorname{char} \mathbb{F} = p > 0$, $P \leq \mathrm{GL}(V)$ a p-group. Can find basis $B = \{x_1, \cdots, x_n\}$ of V^* with $M_B(g)$ upper unitriangular $\forall g \in P$. Define

$$P_x := \mathrm{Stab}_P(x)$$

$N(x) := N^P_{P_x}(x) =$ orbit product of $x \in V^*$,

- $P_i := \cap_{j \neq i} P_{x_j}$

i.e. P_i is "one-column-group" $\begin{pmatrix} 1 & \cdots & 0 & * & 0 & \cdots & 0 \\ 0 & 1 & 0 & * & 0 & \cdots & 0 \\ 0 & 0 & \cdots & * & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

Nakajima groups

Let $\operatorname{char} \mathbb{F} = p > 0$, $P \leq \mathrm{GL}(V)$ a p-group. Can find basis $B = \{x_1, \cdots, x_n\}$ of V^* with $M_B(g)$ upper unitriangular $\forall g \in P$. Define

$$P_x := \mathrm{Stab}_P(x)$$

$$N(x) := N_{P_x}^P = \text{orbit product of } x \in V^*,$$

$$P_i := \cap_{j \neq i} P_{x_j}$$

- P is called a **Nakajima-group** with respect to B, if $P = P_n P_{n-1} \cdots P_1$.

Nakajima groups

Theorem : P is Nakajima-group with respect to $B \iff \mathbb{F}[V]^P = \mathbb{F}[N(x_1), \cdots, N(x_n)]$.

Nakajima groups

Theorem : P is Nakajima-group with respect to $B \iff \mathbb{F}[V]^P = \mathbb{F}[N(x_1), \cdots, N(x_n)]$.

Proof: "$\Rightarrow$": We know (last week!) that $N(x_i)$'s form hsop, moreover $\deg N(x_i) = |P_i|$ with $\prod_i |P_i| = |P|$.

Nakajima groups

Theorem : P is Nakajima-group with respect to $B \iff \mathbb{F}[V]^P = \mathbb{F}[N(x_1), \cdots, N(x_n)]$.

Proof: "$\Rightarrow$": We know (last week!) that $N(x_i)$'s form hsop, moreover $\deg N(x_i) = |P_i|$ with $\prod_i |P_i| = |P|$.

"$\Leftarrow$": A recent theorem of Y.Wu (PhD thesis, Queen's Univ., Kingston 2008).

Nakajima groups

Theorem : P is Nakajima-group with respect to $B \iff \mathbb{F}[V]^P = \mathbb{F}[N(x_1), \cdots, N(x_n)]$.

Proof: "$\Rightarrow$": We know (last week!) that $N(x_i)$'s form hsop, moreover $\deg N(x_i) = |P_i|$ with $\prod_i |P_i| = |P|$.

"$\Leftarrow$": A recent theorem of Y.Wu (PhD thesis, Queen's Univ., Kingston 2008).

Theorem (Nakajima):
If $\mathbb{F} = \mathbb{F}_p$ and $\mathbb{F}[V]^P$ is polynomial ring, then P is Nakajima group.

Nakajima groups

Theorem : P is Nakajima-group with respect to $B \iff \mathbb{F}[V]^P = \mathbb{F}[N(x_1), \cdots, N(x_n)]$.

Proof: "$\Rightarrow$": We know (last week!) that $N(x_i)$'s form hsop, moreover $\deg N(x_i) = |P_i|$ with $\prod_i |P_i| = |P|$.

"$\Leftarrow$": A recent theorem of Y.Wu (PhD thesis, Queen's Univ., Kingston 2008).

Theorem (Nakajima):
If $\mathbb{F} = \mathbb{F}_p$ and $\mathbb{F}[V]^P$ is polynomial ring, then P is Nakajima group.

False for $\mathbb{F}_{p^n}$ $n > 1$ (Counterexample by Stong.)

Nakajima groups

Theorem : P is Nakajima-group with respect to $B \iff \mathbb{F}[V]^P = \mathbb{F}[N(x_1), \cdots, N(x_n)]$.

Proof: "⇒": We know (last week!) that $N(x_i)$'s form hsop, moreover $\deg N(x_i) = |P_i|$ with $\prod_i |P_i| = |P|$.

"⇐": A recent theorem of Y.Wu (PhD thesis, Queen's Univ., Kingston 2008).

Theorem (Nakajima):
If $\mathbb{F} = \mathbb{F}_p$ and $\mathbb{F}[V]^P$ is polynomial ring, then P is Nakajima group.

False for $\mathbb{F}_{p^n}$ $n > 1$ (Counterexample by Stong.)

Kemper/Malle : Classification of all irreducible $G \leq \mathrm{GL}(V)$ with $\mathbb{F}[V]^G$ polynomial ring.

Nakajima groups

Theorem : P is Nakajima-group with respect to $B \iff$ $\mathbb{F}[V]^P = \mathbb{F}[N(x_1), \cdots, N(x_n)]$.

Proof: "$\Rightarrow$": We know (last week!) that $N(x_i)$'s form hsop, moreover $\deg N(x_i) = |P_i|$ with $\prod_i |P_i| = |P|$.

"$\Leftarrow$": A recent theorem of Y.Wu (PhD thesis, Queen's Univ., Kingston 2008).

Theorem (Nakajima):
If $\mathbb{F} = \mathbb{F}_p$ and $\mathbb{F}[V]^P$ is polynomial ring, then P is Nakajima group.

False for $\mathbb{F}_{p^n}$ $n > 1$ (Counterexample by Stong.)

Kemper/Malle : Classification of all irreducible $G \leq \mathrm{GL}(V)$ with $\mathbb{F}[V]^G$ polynomial ring.

Open problem : Characterization of modular representations of finite groups with polynomial rings of invariants.

Non-linear group actions

Theorem : (Fl.-Woodcock) Let $|P| = p^n$ with $p = \mathrm{char}\,\mathbb{F}$. There is a non-linear faithful action on $A := \mathbb{F}[X_1, \cdots, X_n]$ with polynomial ring A^P.

Non-linear group actions

Theorem : (Fl.-Woodcock) Let $|P| = p^n$ with $p = \mathrm{char}\, \mathbb{F}$. There is a non-linear faithful action on $A := \mathbb{F}[X_1, \cdots, X_n]$ with polynomial ring A^P.

A is a retract (i.e. direct summand with ideal complement) of a localization of $\mathbb{F}[V_{\mathrm{reg}}]$.

Non-linear group actions

Theorem : (Fl.-Woodcock) Let $|P| = p^n$ with $p = \mathrm{char}\,\mathbb{F}$. There is a non-linear faithful action on $A := \mathbb{F}[X_1, \cdots, X_n]$ with polynomial ring A^P.

A is a retract (i.e. direct summand with ideal complement) of a localization of $\mathbb{F}[V_{\mathrm{reg}}]$.

Algebra A can be used to obtain a *structure theorem* for Galois-ring extensions with group P.

UFD

Theorem : Let A be UFD, G a finite group acting on A such that

- G acts trivially on A^*
- $\mathrm{Hom}(G, A^*) = 1$

(e.g. $A^* = A_0 = \mathbb{F}$ and $G = [G, G]$ or $|G| = p^k$, $p = \mathrm{char}\ \mathbb{F}$.)

UFD

Theorem : Let A be UFD, G a finite group acting on A such that

- G acts trivially on A^*
- $\mathrm{Hom}(G, A^*) = 1$

Then A^G is a UFD.

UFD

Theorem : Let A be UFD, G a finite group acting on A such that

- G acts trivially on A^*
- $\mathrm{Hom}(G, A^*) = 1$

Then A^G is a UFD.

Proof: Let $\mathfrak{p} \in \mathrm{Spec}_1(A^G)$; need to show : $\mathfrak{p}$ is principal ideal.
know : $\exists \mathfrak{P} \in \mathrm{Spec}_1(A)$ with $\mathfrak{P} \cap A^G = \mathfrak{p}$ ("lying over").
know : $\mathfrak{P} = fA$ with prime $f \in A$.

UFD

Theorem : Let A be UFD, G a finite group acting on A such that

- G acts trivially on A^*
- $\mathrm{Hom}(G, A^*) = 1$

Then A^G is a UFD.

Proof: Let $H := \{g \in G \mid {}^g f \sim f \text{ are associated}\}$, then

$\forall g \in H : \ {}^g f = c_g f \text{ with } c_g \in A^*;$

$H \leq G$ (subgroup).

UFD

Theorem : Let A be UFD, G a finite group acting on A such that

- G acts trivially on A^*
- $\mathrm{Hom}(G, A^*) = 1$

Then A^G is a UFD.

Proof: Let $H := \{g \in G \mid {}^g f \sim f \text{ are associated}\}$, then

$\forall g \in H : \; {}^g f = c_g f$ with $c_g \in A^*$;

$H \leq G$ (subgroup).

Let $G := \biguplus_{r \in \mathcal{R}} rH$ and $N_f := \prod_{r \in \mathcal{R}} {}^r f$, then

$${}^g N_f = \lambda(g) N_f \text{ with } \lambda \in \mathrm{Hom}(G, A^*) = 1.$$

Hence $N_f \in A^G$.

UFD

Theorem : Let A be UFD, G a finite group acting on A such that

- G acts trivially on A^*
- $\mathrm{Hom}(G, A^*) = 1$

Then A^G is a UFD.

Proof: If $\phi \in \mathrm{p}$, then $f \mid \phi \Rightarrow {}^r f \mid \phi \; \forall r \in \mathcal{R} \; \Rightarrow$
$N_f \mid \phi$, so $N_f \cdot a = \phi$ with $a \in A \cap \mathrm{Quot}(A^G) = A^G$.
It follows that $\phi \in N_f \cdot A^G \Rightarrow \mathrm{p} \subseteq N_f \cdot A^G$.

UFD

Theorem : Let A be UFD, G a finite group acting on A such that

- G acts trivially on A^*
- $\mathrm{Hom}(G, A^*) = 1$

Then A^G is a UFD.

Proof: If $\phi \in \mathrm{p}$, then $f \mid \phi \Rightarrow {}^r f \mid \phi \, \forall r \in \mathcal{R} \Rightarrow$
$N_f \mid \phi$, so $N_f \cdot a = \phi$ with $a \in A \cap \mathrm{Quot}(A^G) = A^G$.
It follows that $\phi \in N_f \cdot A^G \Rightarrow \mathrm{p} \subseteq N_f \cdot A^G$.
Clearly $N_f A^G \subseteq f A \cap A^G = P \cap A^G = \mathrm{p}$, hence

$$\mathrm{p} = N_f A^G.$$

Nakajima's theorem

Let $\mathbb{K} := \mathrm{Quot}(A)$; $\mathcal{F}_A :=$ fractional ideals (i.e. fin. gen. $J \leq {}_A\mathbb{K}$).
For $I, J \in \mathcal{F}_A$: $I \sim_{Artin} J \iff I^{-1} = J^{-1}$ (equivalence relation)

$$\mathcal{D}_A := \mathcal{F}_A / \sim_{Artin} \cong \oplus_{\mathrm{p} \in \mathrm{Spec}_1(A)} \mathbb{Z} \cdot d(\mathrm{p}) \text{ (free abelian group)}.$$

Let $\mathcal{H}_A := \{[xA] \mid x \in \mathbb{K}\} \leq \mathcal{D}_A$ (principal fractional ideals)

$$\mathcal{C}_A := \mathcal{D}_A / \mathcal{H}_A = \text{class group, measures distance from UFD}:$$

$$A \text{ is UFD} \iff \mathcal{C}_A = 0.$$

Nakajima's theorem

Let $\mathbb{K} := \mathrm{Quot}(A)$; $\mathcal{F}_A :=$ fractional ideals (i.e. fin. gen. $J \leq {}_A\mathbb{K}$).
For $I, J \in \mathcal{F}_A$: $I \sim_{Artin} J \iff I^{-1} = J^{-1}$ (equivalence relation)

$$\mathcal{D}_A := \mathcal{F}_A / \sim_{Artin} \cong \oplus_{\mathfrak{p} \in \mathrm{Spec}_1(A)} \mathbb{Z} \cdot d(\mathfrak{p}) \text{ (free abelian group).}$$

Let $\mathcal{H}_A := \{[xA] \mid x \in \mathbb{K}\} \leq \mathcal{D}_A$ (principal fractional ideals)

$$\mathcal{C}_A := \mathcal{D}_A / \mathcal{H}_A = \text{class group, measures distance from UFD :}$$

$$A \text{ is UFD} \iff \mathcal{C}_A = 0.$$

Theorem Let $G \leq \mathrm{GL}(V)$ with $W := \langle r \mid r \text{ reflection in } G \rangle \trianglelefteq G$.
Then

$$\mathcal{C}_{A^G} \cong \{\rho \in \mathrm{Hom}(G, \mathbb{F}^*) \mid \mathrm{res}_W(\rho) = 1\}.$$

Non - CM invariant rings

(Campbell,Hughes,Kemper,Shank,Wehlau)
Assume $N \triangleleft G$ with $G/N = \langle \bar{g} \rangle \cong \mathbb{Z}/p$, $p = \mathrm{char}\ \mathbb{F}$.

Theorem For $m \geq 3$: $\mathbb{F}[V^m]^G$ **not** Cohen - Macaulay.

Non - CM invariant rings

(Campbell,Hughes,Kemper,Shank,Wehlau)
Assume $N \triangleleft G$ with $G/N = \langle \bar{g} \rangle \cong \mathbb{Z}/p$, $p = \text{char } \mathbb{F}$.

Theorem For $m \geq 3$: $\mathbb{F}[V^m]^G$ **not** Cohen - Macaulay.

Sketch of proof : $0 \neq \mathcal{J} := (\bar{g} - 1)A^N \cap A^G \triangleleft A^G$

Non - CM invariant rings

(Campbell,Hughes,Kemper,Shank,Wehlau)
Assume $N \triangleleft G$ with $G/N = \langle \bar{g} \rangle \cong \mathbb{Z}/p$, $p = \text{char}\, \mathbb{F}$.

Theorem For $m \geq 3$: $\mathbb{F}[V^m]^G$ **not** Cohen - Macaulay.

Sketch of proof : $0 \neq \mathcal{J} := (\bar{g} - 1)A^N \cap A^G \triangleleft A^G$

Choose $f_1, f_2, f_3 \in \mathcal{J} \Rightarrow f_i = \bar{g}(a_i) - a_i$, $a_i \in A^N$

Non - CM invariant rings

(Campbell,Hughes,Kemper,Shank,Wehlau)
Assume $N \triangleleft G$ with $G/N = \langle \bar{g} \rangle \cong \mathbb{Z}/p$, $p = \text{char}\ \mathbb{F}$.

Theorem For $m \geq 3$: $\mathbb{F}[V^m]^G$ **not** Cohen - Macaulay.

Sketch of proof : $0 \neq \mathcal{J} := (\bar{g} - 1)A^N \cap A^G \triangleleft A^G$

Choose $f_1, f_2, f_3 \in \mathcal{J} \Rightarrow f_i = \bar{g}(a_i) - a_i$, $a_i \in A^N$

$$\Rightarrow 0 = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1 & f_2 & f_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = u_{23} f_1 + u_{31} f_2 + u_{12} f_3,$$

$u_{ij} := \begin{vmatrix} f_i & f_j \\ a_i & a_j \end{vmatrix} \in A^G \backslash (f_i, f_j) A^G$. $\Rightarrow (f_1, f_2, f_3)$ not a regular sequence in A^G. But for $m \geq 3$, J contains partial hsop of length ≥ 3 and in CM-ring, partial hsops are regular sequences.

Modular CM invariant rings

An element $g \in \mathrm{GL}(V)$ is called a **bi-reflection**, if $\mathrm{rk}(g - \mathrm{id}_V) \leq 2$.

Theorem (Kemper): Let $P \leq \mathrm{GL}(V)$ be a p-group ($p = \mathrm{char}\,\mathbb{F}$) such that $\mathbb{F}[V]^P$ is Cohen-Macaulay, then P is generated by bi-reflections.

Modular CM invariant rings

An element $g \in \mathrm{GL}(V)$ is called a **bi-reflection**, if $\mathrm{rk}(g - \mathrm{id}_V) \leq 2$.

Theorem (Kemper): Let $P \leq \mathrm{GL}(V)$ be a p-group ($p = \mathrm{char}\ \mathbb{F}$) such that $\mathbb{F}[V]^P$ is Cohen-Macaulay, then P is generated by bi-reflections.

Corollary : $\mathbb{F}[V^{\oplus 3}]^P$ is not Cohen-Macaulay.

General Question

How close is a A^G to being a Cohen-Macaulay ring ?

Recall : measured by $\operatorname{depth}(A^G) =$

General Question

How close is a A^G to being a Cohen-Macaulay ring ?

Recall : measured by $\mathrm{depth}(A^G) =$

- length of maximal regular sequence in A^G;
- cohomological co - dimension as module over parameter subalgebra $\mathcal{F}$:

$$\mathrm{proj.dim}\ _{\mathcal{F}}A^G + \mathrm{depth}(A^G) = \mathrm{Dim}(A^G)$$

(Auslander-Buchsbaum-formula)

General Question

How close is a A^G to being a Cohen-Macaulay ring ?

Recall : measured by $\mathrm{depth}(A^G) =$

- length of maximal regular sequence in A^G;

- cohomological co - dimension as module over parameter subalgebra $\mathcal{F}$:

$$\mathrm{proj.dim}\,_{\mathcal{F}} A^G + \mathrm{depth}(A^G) = \mathrm{Dim}(A^G)$$

(Auslander-Buchsbaum-formula)

$$\boxed{\mathrm{depth}(A^G) = \mathrm{Dim}(A^G) \iff A^G \text{ CM.}}$$

General Question

How close is a A^G to being a Cohen-Macaulay ring ?

Lemma: Let $C \leq B$ graded connected over $\mathbb{F}$ with ${}_C B$ finite and ${}_C B = C \oplus X$. Then $\mathrm{depth}(B) \leq \mathrm{depth}(C)$.

General Question

How close is a A^G to being a Cohen-Macaulay ring ?

Lemma: Let $C \leq B$ graded connected over $\mathbb{F}$ with ${}_C B$ finite and ${}_C B = \ C \oplus X$. Then $\mathrm{depth}(B) \leq \mathrm{depth}(C)$.

If $H \leq G$ with $[G : H]1_{\mathbb{F}} \neq 0$, then *relative* trace:

$$t_H^G : \ A^H \to A^G, a \mapsto \sum_{g \in G/H} g(a),$$

$A^G \mid {}_{A^G} A^H$, hence $\mathrm{depth}\ A^H \leq \mathrm{depth}\ A^G$.

Let $P \in \mathrm{Syl}_p(G)$: If A^P is CM, so is A^G.

General Question

How close is a A^G to being a Cohen-Macaulay ring ?

Lemma: Let $C \leq B$ graded connected over $\mathbb{F}$ with ${}_C B$ finite and ${}_C B = \; C \oplus X$. Then $\mathrm{depth}(B) \leq \mathrm{depth}(C)$.

If $H \leq G$ with $[G : H]1_{\mathbb{F}} \neq 0$, then *relative* trace:

$$t_H^G : \; A^H \to A^G, a \mapsto \sum_{g \in G/H} g(a),$$

$A^G \mid {}_{A^G} A^H$, hence $\mathrm{depth}\, A^H \leq \mathrm{depth}\, A^G$.

Let $P \in \mathrm{Syl}_p(G)$: If A^P is CM, so is A^G.

However : $\mathbb{F}[V]^{\Sigma_p}$ poly, but $\mathbb{F}[V]^P$ not CM for $P \in \mathrm{Syl}_p(\Sigma_p)$, $p \geq 5$.

The relative transfer ideal

Let $\mathbb{F} = \overline{\mathbb{F}}$ and fix $P \leq G \in \mathrm{Syl}_p(G)$.

$$I_P^G := \sum_{Q<P} t_Q^G(A^Q) \trianglelefteq A^G.$$

- $\mathfrak{i} := \sqrt{I_P^G} \in \mathrm{Spec}(A^G)$ prime ideal of height $\mathrm{codim}(V^P)$;

The relative transfer ideal

Let $\mathbb{F} = \overline{\mathbb{F}}$ and fix $P \le G \in \mathrm{Syl}_p(G)$.

$$I_P^G := \sum_{Q<P} t_Q^G(A^Q) \trianglelefteq A^G.$$

- $\mathfrak{i} := \sqrt{I_P^G} \in \mathrm{Spec}(A^G)$ prime ideal of height $\mathrm{codim}(V^P)$;
- The quotient $A^G/\mathfrak{i}$ is CM of Krull - Dimension $\dim_{\mathbb{F}} V^P$. (Fl. 1998,2002)

The relative transfer ideal

Let $\mathbb{F} = \overline{\mathbb{F}}$ and fix $P \leq G \in \mathrm{Syl}_p(G)$.

$$I_P^G := \sum_{Q<P} t_Q^G(A^Q) \trianglelefteq A^G.$$

- $\mathfrak{i} := \sqrt{I_P^G} \in \mathrm{Spec}(A^G)$ prime ideal of height $\mathrm{codim}(V^P)$;
- The quotient $A^G/\mathfrak{i}$ is CM of Krull - Dimension $\dim_{\mathbb{F}} V^P$. (Fl. 1998,2002)
- $\mathrm{depth}\, A^G = \mathrm{grade}(\mathfrak{i}, A^G) + \dim V^P$; (Fl., Shank 2000) where $\mathrm{grade}(\mathfrak{i}, A^G) :=$ length of a max. reg. sequence in $\mathfrak{i}$.

Calculation of $\mathrm{grade}(\mathfrak{i}, A^G)$

Using D Rees' definition

$$\mathrm{grade}(\mathfrak{i}, A^G) = \min\{k \mid \mathrm{Ext}^k_{A^G}(A^G/\mathfrak{i}, A^G) \neq 0\} \leq h := \mathrm{ht}(\mathfrak{i}).$$

$A^G = H^0(G, A) \Rightarrow$

one can approach computation of $\mathrm{grade}(\mathfrak{i}, A^G)$ via Ellingsrud - Skjelbred spectral sequence (1980):

$$\boxed{\mathrm{E}_2^{p,q} = \mathrm{Ext}^p_{A^G}(A^G/\mathfrak{i}, H^q(G, A)) \Rightarrow H^n(T),}$$

with $H^n(T) = 0$, for $n = p + q < h$.

Minimal depth and cohomology

(with G Kemper, RJ Shank, J Elmer)

$c := \min\{k > 0 \mid H^k(G, A) \neq 0\}$ =: "cohomological connectivity"

Minimal depth and cohomology

(with G Kemper, RJ Shank, J Elmer)

$c := \min\{k > 0 \mid H^k(G, A) \neq 0\}$ =: "cohomological connectivity"

Theorem (Fl.-Kemper-Shank)

$\mathrm{depth}(A^G) \geq \min\{\dim V^P + c + 1, \dim V\};$

Minimal depth and cohomology

(with G Kemper, RJ Shank, J Elmer)

$c := \min\{k > 0 \mid H^k(G, A) \neq 0\}$ =: "cohomological connectivity"

Theorem (Fl.-Kemper-Shank)

$\mathrm{depth}(A^G) \geq \min\{\dim V^P + c + 1, \dim V\}$;

- equality, if and only if $\mathfrak{i} \cdot \alpha = 0$ for some $0 \neq \alpha \in H^c(G, A)$.

$\iff$ $\mathfrak{i}$ is associated prime of $H^c(G, A)$.

Minimal depth and cohomology

(with G Kemper, RJ Shank, J Elmer)

$c := \min\{k > 0 \mid H^k(G, A) \neq 0\}$ =: "cohomological connectivity"

Theorem (Fl.-Kemper-Shank)

$\mathrm{depth}(A^G) \geq \min\{\dim V^P + c + 1, \dim V\}$;

- equality, if and only if $\mathfrak{i} \cdot \alpha = 0$ for some $0 \neq \alpha \in H^c(G, A)$.

$\iff$ $\mathfrak{i}$ is associated prime of $H^c(G, A)$.

Criterion: $\exists 0 \neq \alpha \in H^c(G, A)$ with $\mathrm{res}^G_Q(\alpha) = 0 \, \forall \, Q < P \Rightarrow \mathrm{depth}(A^G)$ minimal.

(J. Elmer (2008)): "$\Leftarrow$" is also true.

Minimal depth and cohomology

(with G Kemper, RJ Shank, J Elmer)

$c := \min\,\{k > 0 \mid H^k(G, A) \neq 0\}$ =: "cohomological connectivity"

Theorem

$$\operatorname{depth}(A^G) = \min\{\dim V^P + c + 1, \dim V\}$$

- if G is p - nilpotent with cyclic $P \in \operatorname{Syl}_p(G)$;
- for $G = P$ cyclic: Ellingsrud-Skjelbred (1980);
- for infinite families of $\mathbb{Z}_p \times \mathbb{Z}_p$-representations including
- all non-projective indecomposable representations of

  $\mathbb{Z}_2 \times \mathbb{Z}_2$ in $\operatorname{char}(\mathbb{F}) = 2$.

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

V is indecomposable if $V = U \oplus W \Rightarrow U = 0$ or $W = 0$.

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

V is indecomposable if $V = U \oplus W \Rightarrow U = 0$ or $W = 0$.

- In non-modular case: "irreducible" $\iff$ "indecomposable".
  In modular case: "irreducible" $\Rightarrow, \not\Leftarrow$ "indecomposable".

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

V is indecomposable if $V = U \oplus W \Rightarrow U = 0$ or $W = 0$.

- In non-modular case: "irreducible" $\iff$ "indecomposable".
  In modular case: "irreducible" $\Rightarrow, \not\Leftarrow$ "indecomposable".

  Example: $G = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_p\}$, $\mathbb{F}_p^2$ is indecomposable but not irreducible.

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

V is indecomposable if $V = U \oplus W \Rightarrow U = 0$ or $W = 0$.

- In non-modular case: "irreducible" $\iff$ "indecomposable". In modular case: "irreducible" $\Rightarrow, \not\Leftarrow$ "indecomposable".

  Example: $G = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_p\}$, $\mathbb{F}_p^2$ is indecomposable but not irreducible.

  $\#$ of iso-types of irreducibles is finite.

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

V is indecomposable if $V = U \oplus W \Rightarrow U = 0$ or $W = 0$.

- In non-modular case: "irreducible" $\iff$ "indecomposable".
  In modular case: "irreducible" $\Rightarrow, \not\Leftarrow$ "indecomposable".

  Example: $G = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_p \}$, $\mathbb{F}_p^2$ is indecomposable but not irreducible.

  $\#$ of iso-types of irreducibles is finite.

- In modular case: $\#$ of iso-types of indecomposables "generically wild"
  (=infinite and "unclassifiable".)

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

Theorem [Karaguezian-Symonds] The number of isomorphism types of indecomposable $\mathbb{F}G$-modules occurring as *direct summand* of $A := \mathbb{F}[V]$ is finite .

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

> Theorem [Karaguezian-Symonds] The number of isomorphism types of indecomposable $\mathbb{F}G$-modules occurring as *direct summand* of $A := \mathbb{F}[V]$ is finite .
>
> (very rough) outline of proof...

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

Theorem [Karaguezian-Symonds] The number of isomorphism types of indecomposable $\mathbb{F}G$-modules occurring as *direct summand* of $A := \mathbb{F}[V]$ is finite .

(very rough) outline of proof...

Let $P \in \mathrm{Syl}_p(G)$, then for every $W \in \mathbb{F}G - \texttt{mod}$,
$W \mid (W_{\downarrow P})^{\uparrow G}$. $\Rightarrow$

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \mathtt{mod}$. Then

Theorem [Karaguezian-Symonds] The number of isomorphism types of indecomposable $\mathbb{F}G$-modules occurring as *direct summand* of $A := \mathbb{F}[V]$ is finite .

(very rough) outline of proof...

Let $P \in \mathrm{Syl}_p(G)$, then for every $W \in \mathbb{F}G - \mathtt{mod}$, $W \mid (W_{\downarrow P})^{\uparrow G}$. $\Rightarrow$

- Reduction to $P \leq P_n(V) := \mathrm{Syl}_p(\mathrm{GL}(V))$.

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

Theorem [Karaguezian-Symonds] The number of isomorphism types of indecomposable $\mathbb{F}G$-modules occurring as *direct summand* of $A := \mathbb{F}[V]$ is finite .

(very rough) outline of proof...

Let $P \in \mathrm{Syl}_p(G)$, then for every $W \in \mathbb{F}G - \texttt{mod}$, $W \mid (W_{\downarrow P})^{\uparrow G}$. $\Rightarrow$

- Reduction to $P \leq P_n(V) := \mathrm{Syl}_p(\mathrm{GL}(V))$.

Use the fact that $A^{P_n(V)} = \mathbb{F}_q[b_1, \cdots, b_n]$ is polynomial.

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \mathtt{mod}$. Then

Theorem [Karaguezian-Symonds] The number of isomorphism types of indecomposable $\mathbb{F}G$-modules occurring as *direct summand* of $A := \mathbb{F}[V]$ is finite .

(very rough) outline of proof...

Let $P \in \mathrm{Syl}_p(G)$, then for every $W \in \mathbb{F}G - \mathtt{mod}$, $W \mid (W_{\downarrow P})^{\uparrow G}$. $\Rightarrow$

- Reduction to $P \leq P_n(V) := \mathrm{Syl}_p(\mathrm{GL}(V))$.

  Use the fact that $A^{P_n(V)} = \mathbb{F}_q[b_1, \cdots, b_n]$ is polynomial.

  Direct summands in low degree are "propagated" into higher degree by multiplication with homogeneous invariants.

Module structure of $\mathbb{F}[V]$

Let $V \in \mathbb{F}G - \texttt{mod}$. Then

Theorem [Karaguezian-Symonds] The number of isomorphism types of indecomposable $\mathbb{F}G$-modules occurring as *direct summand* of $A := \mathbb{F}[V]$ is finite .

(very rough) outline of proof...

Let $P \in \mathrm{Syl}_p(G)$, then for every $W \in \mathbb{F}G - \texttt{mod}$, $W \mid (W_{\downarrow P})^{\uparrow G}$. $\Rightarrow$

- Reduction to $P \leq P_n(V) := \mathrm{Syl}_p(\mathrm{GL}(V))$.

  Use the fact that $A^{P_n(V)} = \mathbb{F}_q[b_1, \cdots, b_n]$ is polynomial.

  Direct summands in low degree are "propagated" into higher degree by multiplication with homogeneous invariants.

  Method+results used to obtain...

General modular degree bound

...long conjectured, now a Theorem of P Symonds (2009): For arbitrary finite $G \leq \mathrm{GL}(V)$:

$$\beta(\mathbb{F}[V]^G) \leq \max\{|G|,\ \dim(V)\cdot(|G|-1)\}.$$

General modular degree bound

...long conjectured, now a Theorem of P Symonds (2009): For arbitrary finite $G \leq \mathrm{GL}(V)$:

$$\beta(\mathbb{F}[V]^G) \leq \max\{|G|,\ \dim(V)\cdot(|G|-1)\}.$$

However, usually not sharp e.g.:

$$\beta(\mathbb{F}_p[V]^{\mathbb{Z}/p}) = (p-1)\dim(V^{\mathbb{Z}/p}) + p - 2\ \ (generically),$$

(former $2p-3$-conjecture for V_{reg} (Fl., Sezer, Shank, Woodcock, 2006)).