1. Use the mapping properties of the $\wp$-function to show that every complex torus $X(l_1, l_2)$ has a biholomorphic automorphism of order 2 with 4 fixed points corresponding to the 4 points of order 2 on the torus, $\wp$-images (denoted $e_0, e_1, \ldots$) of the points $0, \omega_1 = l_1/2); \omega_2 = l_2/2, \omega_3 = (l_1 + l_2)/2.$

Normalising so that $\tau = l_2/l_1$, define the $\lambda$-function as the cross-ratio

$$l(\tau) = \{ e_1, e_2; e_3, e_0 \} = \frac{e_3 - e_1}{e_3 - e_2}.$$

Show that under modular transformations $\tau \mapsto \gamma(\tau) = \frac{a\tau + b}{c\tau + d} \in \Gamma(1)$, the four functions $e_j(\gamma(\tau))$ are a permutation of the $e_j(\tau)$. By evaluating the permutation for the generators $T, U$ of $\Gamma(1)$, show that the $\lambda$-function is invariant under the subgroup consisting of $\gamma \cong 1d \mod 2$ in the congruence subgroup $\Gamma(2)$.

2. Look this function up in the book by Jones & Singerman, pp. 293-296.

3. Show that the congruence subgroup $\Gamma(2)$ is distinct from the commutator subgroup $H(1)$ of $\Gamma(1)$ although they have the same index in $\Gamma(1)$.

What can you discover about the quotient surface $U/H(1)$?

Weekly course summary:

Modular group/ automorphic forms (Bill Harvey, KCL).

Week 1. The upper half plane, Moebius maps and hyperbolic plane geometry.

Week 2. Action of $\text{SL}(2,\mathbb{Z})$ as hyperbolic isometries.

Week 3. Lattices, elliptic functions and Eisenstein series.

Summary notes for these are posted on the LTCC website.


Notes on this are still in preparation.

Week 5. $q$-expansions and related topics; time permitting, mention quadratic forms & theta functions, modular varieties and the Grothendieck-Belyi theory of arithmetically defined algebraic curves.