London Taught Course on Spectral Theory
Problems for weeks 1 – 3.

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The following problems are similar to those that might be asked in the question that I am expected to produce for a final test. You may send me written work, but please only do so when you do not feel confident about your solution for a problem. Also make sure that what you send me is easily readable.

1. Let \( \mathcal{B} = C[0,a] \) and define the Volterra operator \( V \) by

\[
(Vf)(x) = \int_0^x f(s) \, ds.
\]

Prove that \( V \) is a bounded operator and that it has no eigenvalues. Prove also that \( 0 \in \text{Spec}(V) \).

2. Prove that if \( A \) is a bounded operator on \( \mathcal{B} \) and \( \lambda, \mu \notin \text{Spec}(A) \) then

\[
R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A),
\]

\[
R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A).
\]

3. Let \( \theta \in \mathbb{R} \) and let \( \mathcal{H} = L^2(0, 2\pi) \). Define the operator \( A \) by \( (Af)(x) = if'(x) \) on the domain

\[
\mathcal{D} = \{ f \in C^1[0, 2\pi] : f(2\pi) = e^{i\theta} f(0) \}.
\]

Find all of the eigenvalues of \( A \) and use Lemma 1.2.2 to prove that \( A \) is essentially self-adjoint on \( \mathcal{D} \).

4. Let \( \mathcal{H} = L^2(0, 1) \) and define \( A \) by

\[
(Af)(x) = a(x)f''(x) + b(x)f'(x) + c(x)f(x)
\]
on the domain
\[ D = \{ f \in C^2[0,1] : f(0) = f(1) = 0 \}. \]

Assuming that \( a, b, c \) are all sufficiently differentiable, real-valued functions on \([0,1]\), use integration by parts to find the precise conditions that ensure that \( A \) is a symmetric operator.

5. Let \( \mathcal{H} \) be a Hilbert space and let \( P : \mathcal{H} \to \mathcal{H} \) be a bounded operator satisfying \( P = P^* = p^2 \). Prove that \( \mathcal{L} = \text{Ker}(P) \) and \( \mathcal{M} = \text{Ran}(P) \) are orthogonal linear subspaces and that \( \mathcal{H} = \mathcal{L} + \mathcal{M} \) as an algebraic direct sum.

6. Let \( \mathcal{H} = L^2(0,1) \) and define \( A \) by \( (Af)(x) = a(x)f(x) \) where \( a : [0,1] \to \mathbb{C} \) is a bounded continuous function. Use the definition of spectrum to determine \( \text{Spec}(A) \). Can \( A \) have any eigenvalues?

7. Let \( \mathcal{H} = \ell^2(\mathbb{N}) \) where \( \mathbb{N} \) is the set of all natural numbers and define the bounded operator \( A \) by

\[ (Af)(n) = \cos(n)f(n) \]

Find the set of all eigenvalues of \( A \) and prove that \( \text{Spec}(A) = [-1,1] \).

8. Let \( \mathcal{H} = \ell^2(\mathbb{N}) \) and define the bounded operators \( A_m \) on \( \mathcal{H} \) by

\[ (A_m f)(n) = \begin{cases} f(n) & \text{if } n \geq m, \\ 0 & \text{otherwise}. \end{cases} \]

Prove that \( \|A_m\| = 1 \) for all \( m \in \mathbb{N} \) but that \( \lim_{m \to \infty} \|A_m f\| = 0 \) for all \( f \in \mathcal{H} \). In other words \( A_m \) converge to 0 strongly but not in norm.

9. Let \( \mathcal{H} = \ell^2(\mathbb{N}) \) and define the bounded ‘left shift’ operator \( L \) by

\[ (Lf)(n) = f(n+1). \]

Prove that \( \|L\| = 1 \) and that every \( \lambda \in \mathbb{C} \) such that \( |\lambda| < 1 \) is an eigenvalue of \( L \). Find \( \text{Spec}(L) \).