\[
F = U - TS
\]

\[
\Rightarrow \quad T = \frac{1}{\beta}
\]

\[
\text{mean energy (internal energy)}
\]

\[
\text{entropy}
\]

**Proof**

\[
\begin{align*}
\mathcal{E} &= F = -\frac{1}{\beta} \log Z(\beta) = -\frac{1}{\beta} \log \sum_i e^{-\beta E_i} \\
t &= U - TS \quad \text{to show} \quad R = r \\
p_i &= \frac{1}{Z} e^{-\beta E_i} \\
\log p_i &= \log \frac{1}{Z} + \log (e^{-\beta E_i}) \\
&= -\log Z - \beta E_i \\
\Rightarrow \quad S &= -\sum_i p_i \log p_i \\
&= + \sum_i p_i (\log Z + \beta E_i) \\
&= \log Z \cdot 1 + \beta \sum_i p_i E_i \\
&= U - \frac{1}{\beta} S
\end{align*}
\]
\[ \psi(p) = -S + \xi + \frac{\beta}{\beta_3} U \]

\[ \frac{1}{\beta_3} \psi(p) = -TS + \xi + U = F + \frac{\Lambda}{\beta_3} \]

\[ \psi \text{ has minimum} \]

\[ \Rightarrow F \text{ has a minimum} \]

\[ \Rightarrow S \text{ has a maximum} \]

Generalized Statistical Mechanics

Start from more general information measure

\[ I(p) = -S(p) = \sum p_i \ln(p_i) \]

Some function

Example: Tsallis entropy
\[ S_q = \frac{1}{q-1} \left( 1 - \sum_{i} \pi_i \right) \]

\[ q \to 1 \Rightarrow S_q \to S = -\sum \pi_i \log \pi_i \]

For this example
\[ h(p_i) = \frac{\pi_i^{q-1} - 1}{q-1} \]

One defines a \(q\)-logarithm as
\[ \log_q (x) = \frac{x^{1/q} - 1}{1/q} \]

[exercise: Show that
\[ \lim_{q \to 1} \log_q (x) = \log x \]

The inverse function of the \(q\)-logarithm is the \(q\)-exponential
\[ \exp_q (x) := \left( 1 + (1-q) x \right)^{\frac{1}{1-q}} \]

\[ q \to 1 \Rightarrow \exp_q (x) \to e^x \]

Now let's do generalized stuff.
\[ \psi[p] = \sum p_i h(p_i) + \alpha \sum p_i + \beta \sum p_i E_i \]

\[ \frac{\partial}{\partial p_i} \psi[p] = 0 \]

\[ \Rightarrow \text{we obtain} \]

\[ h(p_i) + p_i h'(p_i) + \alpha + \beta E_i = 0 \]

\[ f(p_i) \]

\[ f(p_i) = -\alpha - \beta E_i \]

\[ p_i = f^{-1}(-\alpha - \beta E_i) \]

Nonextensive stat. mech.

= generalized stat. mech. based on maximization of Tsallis entropies

Why that name?

Take two independent systems I and II. Then the Tsallis entropy of the
joint system I, II is non-additive:

Theorem

\[ S_{q}^{I,II} = S_{q}^{I} + S_{q}^{II} - (q-1) S_{q}^{I} \cdot S_{q}^{II} \]

Proof

\[ \sum_{i} (p_{i}^{I})^{q} = 1 - (q-1) S_{q}^{I} \quad (1) \]
\[ \sum_{j} (p_{j}^{II})^{q} = 1 - (q-1) S_{q}^{II} \quad (2) \]

\[ \sum_{i,j} p_{ij}^{q} = \sum_{i} (p_{i}^{I})^{q} \cdot \sum_{j} (p_{j}^{II})^{q} \]

\[ = 1 - (q-1) S_{q}^{I,II} \]

eq (1) x eq. (2)

\[ \Rightarrow \sum_{i} (p_{i}^{I})^{q} \cdot \sum_{j} (p_{j}^{II})^{q} \]

\[ = 1 - (q-1) S_{q}^{I} - (q-1) S_{q}^{II} \]

\[ + (q-1)^2 S_{q}^{I} \cdot S_{q}^{II} \]
\[ S_q^{I, II} = S_q^{I} + S_q^{II} - (q-1) S_q^{I} S_q^{II} \]

q.e.d.

At four further (nice) properties of the Tsallis entropy:

- convexity
  \[ S_q = \frac{1}{q-1} (1 - \sum_i p_i^q) \]
  \[ \frac{\partial}{\partial p_i} S_q = -\frac{q}{q-1} p_i^{q-1} \]
  \[ \frac{\partial^2}{\partial p_i \partial p_j} S_q = -q p_i^{q-2} \delta_{ij} \]
  \[ \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases} \]

- stability

Tsallis entropies are Tschebyshe夫 stable, whereas for example Rényi entropies can even be not.

\[ \Rightarrow \text{After the Shannon entropy, the next-best entropy measures are the Tsallis entropies.} \]
\[
\frac{\partial}{\partial p_i} I[p] + \sum_5 \beta_5 M_i 5^5 = 0
\]

take new Tsallis information

For the canonical ensemble

\[
\frac{\partial}{\partial p_i} I^T_q[p] = \frac{q}{q-1} p_i^{q-1}
\]

\[
\Rightarrow \frac{q}{q-1} p_i^{q-1} + \alpha + \beta E_i = 0
\]

\[
\Rightarrow p_i = \frac{1}{Z_q} \left( 1 - \beta (q-1) E_i \right)^{\frac{q}{q-1}}
\]

Today's notation uses \( q' = 2 - q \) and then renames \( q' \to q \)

\[
\Rightarrow p_i = \frac{1}{Z_q} \left( 1 + \beta (q-1) E_i \right)^{-\frac{q}{q-1}}
\]
Further important entropy measures:

**Jaynesian entropy**

\[ S_x = - \sum_i \frac{P_i^{1+x} - P_i^{1-x}}{2x} \]

As \( x \to 0 \) this again reduces to the Shannon entropy.

**Shannon-Mittal entropies**

\[ S_{x,r} = - \sum_i P_i^r \left( \frac{P_i^x - P_i^{-x}}{2x} \right) \]

reduces to -Jaynesian entropy for \( r = x, q = 1 - 2x \)

- Jaynesian entropy

  \[ \text{for } r = 0 \]

- Other entropy

  \[ \text{for } x = \frac{1}{2} \left( q - q^{-1} \right) \]

  \[ r = \frac{1}{2} \left( q + q^{-1} \right) - 1 \]