Using model

Relevant microstates are 'spin configurations'

\((s_1, s_2, \ldots, s_N)\)

\(s_j \in \{-1, 1\}\)

\[ H \{s\} = -J \sum_{\langle i, j \rangle} s_i s_j - B \sum_{i=1}^{N} s_i \]

summation over nearest neighbours on the d-dim lattice

We treat in detail the case \(d=1\)

Interested in free energy per spin in the limit \(N \to \infty\)

\[ f(\beta) = - \lim_{N \to \infty} \frac{1}{\beta N} \log Z(\beta) \]

Useful method to calculate \(f(\beta)\) is the so-called transfer matrix method.
The transfer matrix is

\[ V = \begin{pmatrix} \nu(\pm 1, \pm 1) & \nu(\pm 1, -1) \\ \nu(-1, \pm 1) & \nu(-1, -1) \end{pmatrix} \]

\[ = \begin{pmatrix} e^{\beta (J + B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta (J - B)} \end{pmatrix} \]
Summation over $s_1, ..., s_N$ is equivalent to matrix multiplication of $N$ matrices $V$. Summation over $s_1$ is taking the trace of $V^N$.

$$\Rightarrow \quad Z_N = \text{trace } V^N$$

$V$ is symmetric.

$$\Rightarrow \quad \text{we can find linearly independent eigenvectors } x_j, x_j' \text{ s.t. }$$

$$V \cdot x_j = \lambda j x_j \quad j = 1, 2$$

Trace is invariant under equivalence transformations of a matrix.

$$\Rightarrow \quad Z_N = \text{trace } \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^N = \lambda_1^N + \lambda_2^N$$

$$\lambda_1 > \lambda_2$$

$$f(\beta) = - \lim N \to \infty \frac{1}{\beta N} \log Z_N$$

$$= - \lim N \to \infty \frac{1}{\beta N} \log \left\{ \lambda_1^N \left( 1 + \left( \frac{\lambda_2}{\lambda_1} \right)^N \right) \right\}$$

$$\to 0$$
Hence

\[ F(\beta) = -\lim_{N \to \infty} \frac{1}{N} \frac{1}{\beta} \frac{1}{N} \log \lambda_1 \]

\[ = -\frac{1}{\beta} \log \lambda_1 \]

Very general result:

Free energy given by

\[ \log \text{ of largest eigenvalue of transfer operator} \]
Non-equilibrium systems

Stochastic process $Y(t)$

$Y \in \mathbb{R}^n$

Look at $n$-point functions

$P_n(Y_1, t_1; Y_2, t_2; \ldots; Y_n, t_n)$

$= P_{n-1}(Y_2, t_2; \ldots; Y_n, t_n) \cdot P_{1, n-1}(Y_1, t_1 | Y_2, t_2; \ldots; Y_n, t_n)$

Need all $n$-point functions for $n = 1, 2, 3, \ldots$ to fully characterize stock process.

Important subclasses of stock processes:

Markov process

$P_{n-1}(Y_1, t_1; \ldots; Y_{n-1}, t_{n-1}) = P_{11n}(Y_1, t_1 | Y_2, t_2; \ldots; Y_{n-1}, t_{n-1})$

Wiener process $W(t)$

$P_n(Y_1, t) = \frac{1}{(2\pi t)^{n/2}} \exp \left\{ -\frac{Y_1^2}{2t} \right\}$

$P_{11n}(Y_2, t_2 | Y_1, t_1) = \frac{1}{(2\pi (t_2 - t_1))^{n/2}} \exp \left\{ -\frac{(Y_2 - Y_1)^2}{2(t_2 - t_1)} \right\}$
Stochastic Differential Eq.

\[ \dot{Y} = G(X, Y, Z, \ldots) \]

Solving the stochastic diff. eq. means finding the stochastic process \( Y(t) \) that satisfies it.

Most important stochastic diff. eq. in statistical physics is the Langevin eq.

\[ dY = A(Y) \, dt + \sigma \, dW \]

\[ \iff \quad \dot{Y} = A(Y) + \sigma \, L(t) \]

\[ A : \mathbb{R}^m \rightarrow \mathbb{R}^m \]

\[ \langle L(t) \rangle = 0 \]

\[ \langle L(t_1) L(t_2) \rangle = \delta(t_1 - t_2) \]

\( L(t) \) is a rapidly fluctuating random force.
Example

Brownian particle

\[ A(y) = -y \]

\( y \): velocity of the Brownian particle
\( f \): friction constant

\[ \dot{y} = -f y + \sigma \xi(t) \]

Solution of this stochastic diff. eq. in the Ornstein-Uhlenbeck process

\[ P_0(y, t) = \left( \frac{\pi \sigma^2}{y} \right)^{-\frac{1}{2}} \exp \left\{ -\frac{\sigma^2}{2} \frac{y^2}{t} \right\} \]

\[ P_{10}(y_1, t_2|y_2, t_1) = \frac{e^{-\frac{\sigma^2}{2} \frac{(y_1 - y_2 e^{-\sigma t_{12}})^2}{(1 - e^{-2\sigma t_{12}})}}}{(\pi \sigma^2)^{-\frac{1}{2}} (1 - e^{-2\sigma t_{12}})^{-\frac{1}{2}}} \exp \left\{ -\frac{\sigma^2}{2} \frac{(y_1 - y_2 e^{-\sigma t_{12}})^2}{(1 - e^{-2\sigma t_{12}})} \right\} \]

\[ t := t_2 - t_1 \]

Theorem

The general solution of the Langevin eq. for general \( A(y) \) is a Markov process whose 1-point function is \( P_0(y, t) \) and transition prob. \( P_{10}(y_1, t_2|y_2, t_1) \)
satisfy the \textbf{Fokker-Planck} eq. \( \nabla \)

\[
\frac{\partial P}{\partial t} = -\nabla (AP) + \frac{1}{2} \sigma^2 \Delta P
\]

\( A(Y) : \mathbb{R}^m \rightarrow \mathbb{R}^m \)

\[
Y = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}
\]

\[
\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_m} \end{pmatrix}
\]

\[
\Delta = \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_m^2}
\]

\( P = P_n (y, t) \)

(it if \( P \) is \( P_{1n} \), then \( t = t_n \))

\textbf{Stationary solution of the Fokker-Planck eq.}

\[
\frac{\partial P}{\partial t} = 0
\]

\[
\Rightarrow 0 = -\nabla (AP) + \frac{1}{2} \sigma^2 \Delta P
\]

\( 0 = \text{const} = -AP + \frac{1}{2} \sigma^2 \nabla P \)

\[
\nabla P = \frac{2}{\sigma^2} AP
\]

\[
\frac{\partial}{\partial y_1} P = \frac{2}{\sigma^2} A(Y) P
\]

\[
\vdots
\]

\[
\frac{\partial}{\partial y_m} P = \frac{2}{\sigma^2} A(Y) P
\]
Claim: solution is given by

\[ P(y, x) = \text{const} \cdot \exp \left\{ -\frac{2}{\sigma^2} V(y) \right\} \]

where

\[ A(y) = -\sigma V(y) \]

Proof:

\[ \Delta P(y) = \text{const} \cdot \exp \left\{ -\frac{2}{\sigma^2} V(y) \right\} \cdot \left( -\frac{2}{\sigma^2} \right) \Delta V(y) \]

\[ = P(y) \cdot \frac{2}{\sigma^2} A(y) \]

q.e.d.