We call an equation integrable if it possesses infinitely many higher order generalised symmetries. A recursion operator is a linear operator $\mathcal{R}$ mapping a symmetry to a new symmetry. We can generate infinitely many symmetries by applying it successively to a known symmetry.

**Proposition.** For evolutionary equation $u_t = K$, a linear operator $\mathcal{R}$ is a recursion operator if $\mathcal{R}$ satisfies

$$\mathcal{R}_t = [K_*, \mathcal{R}],$$

where the time derivative $\mathcal{R}_t$ is evaluated on the solution to the equation.

**Proof.** We know that $Q$ is a symmetry if and only if

$$(D_t - K_*)Q = 0.$$ 

If $\mathcal{R}$ satisfies (1) and $G = \mathcal{R}Q$, then

$$(D_t - K_*)(\mathcal{R}Q) = \mathcal{R}_t Q + \mathcal{R}Q_t - K_*(\mathcal{R}Q) = \mathcal{R}(D_t - K_*)Q = 0.$$ 

Thus $G$ is also a symmetry and the statement follows.

**Example.** Show that $\mathcal{R} = D_x + u_1$ is a recursion operator for the potential Burgers equation $u_t = K = u_2 + u_1^2$.

**Proof.** To prove it, we need to check (1). Note that $K_* = D_x^2 + 2u_1 D_x$ and $\mathcal{R}_t = u_{xt} = u_3 + 2u_1 u_2$. By direct computation, we can check $[K_*, \mathcal{R}] = u_3 + 2u_1 u_2$. □
Recursion operators

**Proposition.** Let $\mathcal{R}_1, \mathcal{R}_2$ be two recursion operators for equation $u_t = K$. Then

- $\alpha_1 \mathcal{R}_1 + \alpha_2 \mathcal{R}_2$, $\alpha_1, 2 \in \mathbb{C}$,
- $\mathcal{R}_1 \circ \mathcal{R}_2$ and $\mathcal{R}_2 \circ \mathcal{R}_1$,
- $\mathcal{R}_1^n$ and $\mathcal{R}_2^n$, $n \in \mathbb{N}$,

are also recursion operators.

Let us consider solutions of equation (1) in the space of formal series. The above proposition can be extended.

**Proposition.** Let a formal series $\mathcal{R}$ of order $n$ be a solution of (1) (formal recursion operator). Then

- $\mathcal{R}^\frac{1}{n}$,
- $\mathcal{R}^n$, $n \in \mathbb{Z}$,

are also solutions of (1).

**Example.** Consider the heat equation $u_t = u_2$. Show $\mathcal{R}_1 = D_x$ and $\mathcal{R}_2 = tD_x + \frac{1}{2}x$ are its recursion operators. Can you think a simple symmetry for the equation?
Examples for recursion operators

For majority integrable equations, their recursion operators are weakly nonlocal operators.

▶ The Burgers equation: \( u_t = u_2 + 2uu_1 \)
\[
\mathcal{R} = D_x + u + u_1 D_x^{-1}
\]

▶ The KdV equation: \( u_t = u_3 + uu_1 \)
\[
\mathcal{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}
\]

▶ The Sawada-Kotera equation: \( u_t = u_5 + 5uu_3 + 5u_1 u_2 + 5u^2 u_1 \)
\[
\mathcal{R} = D_x^6 + 6uD_x^4 + 9u_1 D_x^3 + (9u^2 + 11u_2) D_x^2 + (10u_3 + 21uu_1) D_x
+ 5u_4 + 16uu_2 + 6u_1^2 + 4u^3
+(u_5 + 5uu_3 + 5u_1 u_2 + 5u^2 u_1) D_x^{-1} + u_1 D_x^{-1}(2u_2 + u^2)
\]

▶ Nonlinear Schrödinger equation \( u_t = v_2 \mp v(u^2 + v^2); \ v_t = -u_2 \pm u(u^2 + v^2) \)
\[
\mathcal{R} = \left( \begin{array}{cc} \mp 2vD_x^{-1}u & D_x \mp 2vD_x^{-1}v \\ -D_x \pm 2uD_x^{-1}u & \pm 2uD_x^{-1}v \end{array} \right)
= \left( \begin{array}{cc} 0 & D_x \\ -D_x & 0 \end{array} \right) + 2 \left( \begin{array}{c} -v \\ u \end{array} \right) \otimes D_x^{-1}( \pm u \quad \pm v )
Applying the powers of the recursion operator $\mathcal{R}^k$ to a seed symmetry $G_1$ we can, in principle, construct a sequence of symmetries $G_n = \mathcal{R}(G_{n-1})$.

For the Burgers and KdV equations we can take $G_1 = u_1$ as a seed. For the Sawada-Kotera equation there are two seed symmetries:

$$G_1 = u_1, \quad H_1 = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1.$$

The difficulty here is to prove that $G_n = \mathcal{R}(G_{n-1}) \in \text{Dom} \mathcal{R}$. It can be done for so-called hereditary (or Nijenhuis) operators, i.e. operators satisfying the property

$$\mathcal{R}^2[f, g] + [\mathcal{R}(f), \mathcal{R}(g)] - \mathcal{R}([f, \mathcal{R}(g)] + [\mathcal{R}(f), g]) = 0$$

for any two elements $f, g \in \mathcal{A}$.

**Theorem.** Let the nonlocal terms of a Nijenhuis operator $\mathcal{R}$ be of the form

$$\sum_{j=1}^l f_j \otimes D^{-1}x g_j.$$

If for all $j, k = 1, \cdots, l$, we have $f_j \cdot g_k \in \text{Im} D_x$ and both $g_j^*$ and $(\mathcal{R}^\dagger g_j)^*$ are self-adjoint, then for any $h_0$ such that $D_{h_0}(\mathcal{R}) = [h_0^*, \mathcal{R}]$ and $h_0 \cdot \beta_j \in \text{Im} D_x$, all $h_i = \mathcal{R}^i(h_0)$ are local and commute, where $i = 0, 1, 2, \cdots$.

The proof will be given after we introduce the complex over a Lie algebra.
In general, it is not easy to construct a recursion operator for a given integrable equation although the explicit formula is given. The difficulty lies in how to determine the starting terms of $\mathcal{R}$, i.e., the order of the operator, and how to construct its nonlocal terms. Here we list a few references, by no means the list is complete.


If the Lax representation of the equation is known, there is an amazingly simple approach to construct a recursion operator proposed in


This idea has been developed for the Lax pairs that are invariant under the reduction groups

Definition. Let $u_t = K$ be an $n$-th order evolutionary equation. A formal series $D$ of order $m$ is called a formal recursion operator of rank $k$ if

$$\text{ord}(D_t - [K_*, D]) \leq n + m - k.$$ 

Proposition. Suppose equation $u_t = K$ has a symmetry $Q \in A$ of order $m$, then $G_*$ is a formal recursion operator of order $m$ and of rank $m$.

Proof. Taking the Fréchet derivative from the symmetry condition $\frac{\partial Q}{\partial t} + [K, Q] = 0$ we get

$$\frac{\partial Q_*}{\partial t} + (D_K(Q) - D_Q(K))_* = \frac{\partial Q_*}{\partial t} + D_K(Q_*) - [K_*, Q_*] - D_Q(F_*) = 0$$

and thus

$$\text{ord} \left( \frac{\partial Q_*}{\partial t} + D_K(Q_*) - [K_*, Q_*] \right) = \text{ord}D_Q(K_*) \leq \text{ord}K_* = \text{ord}K_* + \text{ord}Q_* - m. \quad \square$$

- The higher the rank of the formal recursion operator, the more restrictive the conditions imposed on the equation.
- The recursion operator is a formal recursion operator of rank $\infty$.
- $D = c$ is trivially a formal recursion operator of rank $\infty$. Integrable equations possess a nonconstant formal recursion operator of rank $\infty$.
- The existence of formal recursion operator of rank $\infty$ provides us with necessary integrability conditions: It is known that a second order evolutionary equation if integrable if and only if it has a formal recursion operator of rank 5 and third order with rank 8.
Using the definition of formal recursion operator, we can prove the following statement (Do it as an exercise!).

**Proposition.** If a formal series $\mathcal{D}$ is a formal recursion operator of rank $k$, then its inverse is also a formal recursion operator of rank $k$. If $\mathcal{D}$ is of order $m \in \mathbb{N}$, then any fractional power $\mathcal{D}^{i/m}$ is also a formal recursion operator of rank $k$.

We can take a fraction power $\mathcal{Q}^\frac{1}{m}$ in order to obtain a formal recursion operator of order 1 and of rank $m$.

**Corollary.** If an evolutionary equation is integrable, it has a first order formal recursion operator of rank $\infty$.

**Example.** For constants $a$ and $b$, equation $u_t = K = u_2 + auu_1 + bu_1^2$ is integrable if and only if $ab = 0$.

**Outline of proof.** The formal recursion operator is of form

$$\mathcal{D} = D_x + Q_0 + Q_1 D_x^{-1} + Q_2 D_x^{-2} + \cdots$$

where $Q_i$ are differential functions to be determined.

Notice that $K_* = D_x^2 + K_1 D_x + K_0 = D_x^2 + (au + 2bu_1)D_x + au_1$. To have a formal recursion operator of rank 3, we require $Q_0 = \frac{1}{2} K_1 = \frac{1}{2} au + bu_1$. To have a formal recursion operator of rank 4, we require

$$D_t Q_0 + D_x K_0 - 2D_x Q_1 - K_1 D_x Q_0 = 0.$$

In order to find $Q_1$, we require that $aK = a(u_2 + auu_1 + bu_1^2)$ lies in the image of $D_x$, which implies $ab = 0$. \qed
Proposition. If $\mathcal{D}$ is a first order formal recursion operator of $n$-th order evolutionary equation $u_t = K$ of rank $k \geq n + 2$, then the residues of the first $k - n - 2$ power of $\mathcal{D}$, that is,

$$\rho_j = \text{res} \mathcal{D}^j, \quad j = 1, \ldots, k - n - 2,$$

are conserved densities.

Proof. Notice that each power of $\mathcal{D}^j$ is a formal recursion operator of rank $k$.

Therefore,

$$\text{ord} (\mathcal{D}^j_t - [K_*, \mathcal{D}^j]) \leq n + j - k.$$ 

If $n + j - k < -1$, the coefficient of $D_x^{-1}$ is zero. According to Adler’s theorem, its residue is of form

$$D_t (\text{res} \mathcal{D}^j) + D_x \sigma_j = 0,$$

which are the required conservation laws. □

We can also show that $\rho_{-1} = \text{res} \mathcal{D}^{-1}$ and $\rho_0 = \text{res} \log \mathcal{D}$ are conserved densities (recall $D_t(\text{res} \log \mathcal{D}) = \text{res} (D_t(\mathcal{D}) \circ \mathcal{D}^{-1})$).

Corollary. Let $\mathcal{R}$ be a recursion operator of order $m$ for equation $u_t = K$. Then

$$\rho_{-1} = \text{res} \mathcal{R}^{-\frac{1}{m}}, \quad \rho_0 = \text{res} \log \mathcal{R}, \quad \rho_1 = \text{res} \mathcal{R}^{\frac{1}{m}}, \quad \ldots, \quad \rho_k = \text{res} \mathcal{R}^{\frac{k}{m}}, \quad \ldots$$

are canonical conserved densities.
Formal recursion operators and Conservation laws

Example. The Burgers equation \( u_t = u_2 + 2uu_1 \) possesses a recursion operator
\[
\mathcal{R} = D_x + u + u_1 D_x^{-1} = D_x \circ (D_x + u) \circ D_x^{-1}.
\]
It generates trivial conserved densities except \( \rho_0 \):
\[
\rho_{-1} = 1, \rho_0 = u, \rho_k = D_x((D_x + u)^{k-1} u) \in \text{Im} D_x
\]

Example. The KdV equation \( u_t = u_3 + uu_1 \) possesses a recursion operator
\[
\mathcal{R} = D_x^2 + \frac{2}{3} u + \frac{1}{3} u_1 D_x^{-1}.
\]
Notice that
\[
\mathcal{R}^{1/2} = D_x + \frac{1}{3} u D_x^{-1} - \frac{1}{18} u^2 D_x^{-3} + \frac{1}{9} uu_x D_x^{-4} + \cdots
\]
Its canonical conserved densities are
\[
\rho_{-1} = 1, \rho_0 = 0, \rho_1 = 2u, \rho_2 = 2u_1, \rho_3 = 2u_2 + u^2, \ldots
\]
It can be shown that \( \text{res}(\mathcal{R}^{2l+1}/2), l = 0, 1, \cdots \) provides the infinite many conserved densities for the KdV equation.

For the remarkable application of formal recursion operators in classification of integrable equations we refer to the following review paper and its references:

Proposition. Let $\mathcal{R}$ be a recursion operator for equation $u_t = K$ and $\gamma$ be its co-symmetry such that $\gamma \in \text{Dom} \mathcal{R}^\dagger$, then $\mathcal{R}^\dagger(\gamma)$ is also a co-symmetry of the equation.

Proof. We know $\gamma_t + K_\ast^\dagger(\gamma) = 0$ and thus

$$D_t(\mathcal{R}^\dagger \gamma) + K_\ast^\dagger(\mathcal{R}^\dagger \gamma) = \mathcal{R}^\dagger_t \gamma + \mathcal{R}^\dagger \gamma_t + K_\ast^\dagger(\mathcal{R}^\dagger \gamma)$$

$$= \left(\mathcal{R}^\dagger_t - \mathcal{R}^\dagger K_\ast^\dagger + K_\ast^\dagger \mathcal{R}^\dagger\right)(\gamma) = (\mathcal{R}^\dagger_t - K_\ast \mathcal{R} + \mathcal{R} K_\ast)^\dagger(\gamma) = 0. \square$$

In this sense as a recursion operator, $\mathcal{R}^\dagger$ is a **co-recursion** operator mapping cosymmetry to a new cosymmetry. It satisfies the condition

$$\mathcal{R}^\dagger_t + [K_\ast^\dagger, \mathcal{R}^\dagger] = 0.$$ 

Example. For the KdV equation $\mathcal{R}^\dagger = D_x^2 + 4u - 2D_x^{-1} \circ u_1$. Taking $\gamma_0 = \frac{1}{2}$ we get

$$\gamma_k = \mathcal{R}^\dagger_k \gamma_0:$$ 

$$\gamma_1 = u, \quad \gamma_2 = u_2 + 3u^2, \quad \gamma_3 = u_4 + 10u_2u + 5u_1^2 + 10u_3, \ldots .$$

Like with the recursion operator, there is the issue of locality, i.e. to show that $\gamma_k \in \text{Dom} \mathcal{R}^\dagger$ which in this case is equivalent to $(\gamma_k)_\ast = (\gamma_k)^\dagger_\ast$.

In the case of the Burgers equation we have $\gamma_0 = 1$, $\mathcal{R}^\dagger = D_x^{-1} \circ (-D_x + u)D_x$ and thus $\mathcal{R}^\dagger \gamma_0 = 0$. The Burgers equation has only one co-symmetry.

In the same way as we defined formal recursion operators, we can define formal co-recursion operators. We won’t investigate it in details here.
What do we mean by an evolutionary equation being a Hamiltonian system?

\[
\frac{dx}{dt} = J(x) \nabla H; \quad \text{↔} \quad u_t = K = \mathcal{D} \delta_u \int h
\]

The Korteweg-de Vries equation can be written as

\[
u_t = u_3 + uu_1 = D_x (u_2 + \frac{u^2}{2}) = D_x \delta_u (\frac{u^3}{6} - \frac{u_1^2}{2})
\]

\[
= (D_x^3 + \frac{2}{3} uD_x + \frac{1}{3} u_1)\delta_u (\frac{u^2}{2})
\]

- the Hamiltonian function \( H \longrightarrow \) a Hamiltonian functional \( \int h \);
- the gradient \( \nabla \longrightarrow \) the variational derivative \( \delta_u \);
- the anti-symmetric matrix \( J(x) \) defining the Poisson bracket

\[
\{ F, G \} = \nabla F \cdot J(x) \nabla G
\]

\[\longrightarrow\] the linear operator \( \mathcal{D} \) defining the Poisson bracket ?

\[
\{ \int f, \int g \} = \int \delta uf \cdot \mathcal{D} \delta ug
\]
Hamiltonian operators and Hamiltonian vector fields

**Definition.** A linear operator $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is called **Hamiltonian** if the above Poisson bracket satisfies the conditions:

- anti-symmetry:
  \[
  \{ \int f, \int g \} = - \{ \int g, \int f \}; \tag{3}
  \]

- the Jacobi identity
  \[
  \{ \{ \int f, \int g \}, \int h \} + \{ \{ \int g, \int h \}, \int f \} + \{ \{ \int h, \int f \}, \int g \} = 0. \tag{4}
  \]

for all functionals $\int f, \int g$ and $\int h$.

**Proposition.** To each functional $\int f$, there is an evolutionary vector field $\mathcal{D}Q$ called a Hamiltonian vector field associated with it satisfying

\[
\mathcal{D}Q \left( \int g \right) = \{ \int g, \int f \}
\]

for all functional $\int g$. Moreover, the characteristic $Q = \mathcal{D}\delta uf$.

**Proof.** Using integration by parts we have

\[
\{ \int g, \int f \} = \int \delta u g \cdot \mathcal{D} \delta uf = \int g_\ast (\mathcal{D} \delta uf) = \int \mathcal{D}_\mathcal{D} \delta uf (g) = \mathcal{D}_\mathcal{D} \delta uf (\int g).
\]

Thus the formula holds and $Q = \mathcal{D} \delta uf$. □

Therefore, the Hamiltonian system of an evolutionary equation takes the form

\[
u_t = \mathcal{D} \delta u \int f.
\]

Hamiltonian equation.
Hamiltonian operators

How to determine whether a given operator $\mathcal{D}$ is Hamiltonian?

**Proposition.** The bracket defined by (2) is anti-symmetric if and only if $\mathcal{D}$ is skew-adjoint: $\mathcal{D}^\dagger = -\mathcal{D}$.

**Proof.** We write (3) as

$$0 = \int \delta_u f \cdot \mathcal{D} \delta_u g + \int \delta_u g \cdot \mathcal{D} \delta_u f = \int \delta_u f \cdot (\mathcal{D} + \mathcal{D}^\dagger) \delta_u g.$$  

Here we draw the conclusion from “substitution principle”: if an expression vanishes for the variational derivatives, it also vanishes for differential functions (Ex 5.42 in Olver’s book).

**Proposition.** Let $\mathcal{D}$ be skew-adjoint. Then the bracket (2) satisfies the Jacobi identity if and only if

$$\int (P \cdot D_{DQ}(\mathcal{D}) R + Q \cdot D_{DR}(\mathcal{D}) P + R \cdot D_{DP}(\mathcal{D}) Q) = 0,$$  

for all $P, Q, R \in \mathcal{A}$.  

(5)
Proof. Let $\delta_u f = P$, $\delta_u g = Q$ and $\delta_u h = R$. We compute

$$\left\{ \left\{ \int f, \int g \right\}, \int h \right\} = D_{DR} \int P \cdot DQ$$

$$= \int (D_{DR} P \cdot DQ + P \cdot D_{DR}(D)Q + P \cdot D(D_{DR} Q))$$

$$= \int P^* (D) \cdot DQ + \int P \cdot D_{DR}(D)Q - \int DP \cdot Q^* (D)$$

Notice that $P^*$, $Q^*$ and $R^*$ are all self-adjoint. We expand the Jacobi identity and we get (5) for all $P, Q, R$ being variational derivatives of functionals. Using “substitution principle”, we complete the proof.

Corollary. Let $\mathcal{D}$ be skew-adjoint. If it doesn’t depend on dependent variables, then $\mathcal{D}$ is a Hamiltonian operator.

We can further simplify the computation by introducing functional multi-vectors.
A **functional** $k$-**vector** is an alternating $k$-linear map from functional one-forms (dual space of the space of evolutionary vector fields) to the space of functionals.

- We use the notation $\theta_i$ for “uni-vector” corresponding to the one-form $du_i$;
- Any functional uni-vector is of canonical form $\int f \theta$, where $f \in \mathcal{A}$;
- Any functional bi-vector has the canonical form
  \[
  \Theta = \frac{1}{2} \int \theta \wedge \mathcal{D} \theta, 
  \]
  where $\mathcal{D}$ is skew-adjoint linear operator. It defines the bilinear map as follows:
  \[
  \Theta(P, Q) = \frac{1}{2} \int (P \cdot \mathcal{D} Q - Q \cdot \mathcal{D} P) = \int P \cdot \mathcal{D} Q 
  \]
- We define a functional tri-vector
  \[
  \Psi = \frac{1}{2} \int \theta \wedge D_{\mathcal{D} \theta}(\mathcal{D}) \wedge \theta \tag{6}
  \]

**Exercise.** Let $\mathcal{D}$ be skew-adjoint. Then $\Psi(P, Q, R)$ is the left-hand side of (5).
**Proposition.** Let $\mathcal{D}$ be skew-adjoint. Then $\mathcal{D}$ is Hamiltonian if and only if the functional tri-vector (6) vanishes: $\Psi = 0$.

**Example.** Show operator $\mathcal{D} = D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_1$ is Hamiltonian.

**Proof.** For this operator, we have

$$D_{D\theta}(\mathcal{D}) = \frac{2}{3} (D\theta) D_x + \frac{1}{3} D_x (D\theta)$$

$$= \frac{2}{3} (\theta_3 + \frac{2}{3} u \theta_1 + \frac{1}{3} u_1 \theta) D_x + \frac{1}{3} (\theta_4 + \frac{2}{3} u \theta_2 + u_1 \theta_1 + \frac{1}{3} u_2 \theta)$$

This leads to

$$\Psi = \int \frac{1}{3} \theta \wedge \theta_3 \wedge \theta_1 = \frac{1}{3} \int D_x (\theta \wedge \theta_1) \wedge \theta_2 = 0. \quad \Box$$

We extend the action of $D_{D\theta}$ to uni-vectors by setting

$$D_{D\theta}(\theta_j) = 0$$

Thus we find that

$$D_{D\theta}(\Theta) = \frac{1}{2} \int D_{D\theta} (\theta \wedge \mathcal{D}\theta) = -\Psi$$

**Theorem.** Let $\mathcal{D}$ be skew-adjoint and $\Theta = \frac{1}{2} \int \theta \wedge \mathcal{D}\theta$ the corresponding functional bi-vector. Then $\mathcal{D}$ is Hamiltonian if and only if $D_{D\theta}(\Theta) = 0$. 


Example. Consider the Bussinesq equation

\[ u_{tt} = u_4 + 4D_x^2(u^2) \]

It can be written as an evolutionary system

\[ u_t = v_1, \quad v_t = u_3 + 8uu_1 \]

It is a Hamiltonian system with the Hamiltonian functional being \( \int \frac{1}{2} v \) and the Hamiltonian operator being

\[
D = \begin{pmatrix}
D_x^3 + 2uD_x + u_1 & 3vD_x + 2v_x \\
3vD_x + v_1 & D_x^5 + 5(uD_x^3 + D_x^3 u) - 3(u_2D_x + D_x u_2) + 16uD_x u
\end{pmatrix}
\]

We will check it using the above Theorem.

Let \( w = (\theta, \eta) \) be the basic uni-vectors corresponding to \( u \) and \( v \) respectively. First we have

\[
\Theta = \frac{1}{2} \int (\theta \wedge \theta_3 + 2u\theta \wedge \theta_1 + 2v\theta \wedge \eta_1 - 4v\theta_1 \wedge \eta + \eta \wedge \eta_5 \\
+ 4u\eta \wedge \eta_3 - 6u\eta_1 \wedge \eta_2 + 16u^2 \eta \wedge \eta_1)
\]

Notice that

\[
D_{Dw}(u) = \theta_3 + 2u\theta_1 + u_1\theta + 3v\eta_1 + 2v_1\eta;
\]

\[
D_{Dw}(v) = 3v\theta_1 + v_1\theta + \eta_5 + 10u\eta_3 + 15u_1\eta_2 + (9u_2 + 16u^2)\eta_1 + (2u_3 + 16uu_1)\eta.
\]

Using this fact and integration by parts, we can prove that \( D_{Dw}(\Theta) = 0 \) (Complete it as an Exercise!)