Analytical Methods exam question 2022 - SOLUTIONS

1. Consider the following partial differential equation for $f(x,y)$:

$$2\frac{(y+1)}{x}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} - f + \lambda\left(\frac{\partial^2 f}{\partial x^2} - 4\frac{\partial^2 f}{\partial y^2}\right) = -x^2 y$$

in the domain $-\infty < x < \infty$, $y \geq 0$. If $\lambda \gg 1$ and the boundary conditions are given by

$$f(x,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x,0) = 2,$$

determine the solution $f(x,y)$ up to and including terms of order $1/\lambda$.

Solution:

A sample solution using the analytical and perturbation techniques introduced in the course is presented below representing what a very good student could do. Attempts to apply other suitable methods will also be marked highly.

We take λ large: for convenience set $\lambda = \varepsilon^{-1}$. We are solving:

$$\frac{\partial^2 f}{\partial x^2} - 4\frac{\partial^2 f}{\partial y^2} + \varepsilon\left(2\frac{(y+1)}{x}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} - f\right) = -\varepsilon x^2 y$$

$$f(x,0) = 0 \qquad \frac{\partial f}{\partial y}(x,0) = 2$$

Leading order

At leading order we have

$$\frac{\partial^2 f}{\partial x^2} - 4\frac{\partial^2 f}{\partial y^2} = 0$$

which is just the homogeneous wave equation. The general solution is

$$f(x,y) = p(2x+y) + q(2x-y)$$

and applying the boundary conditions gives

$$p(2x) + q(2x) = 0 \qquad p'(2x) - q'(2x) = 2$$

$$p(t) = -q(t) = t \qquad f(x,y) = 2y.$$

Order ε

We now put
$$f(x,y) = 2y + \varepsilon f_1(x,y) + \cdots$$
and the governing equation for f_1 becomes
$$\frac{\partial^2 f_1}{\partial x^2} - 4\frac{\partial^2 f_1}{\partial y^2} = 2y + 2 - x^2 y$$
$$f_1(x,0) = 0 \qquad \frac{\partial f_1}{\partial y}(x,0) = 0.$$
The equation is the inhomogeneous wave equation so we can solve it by a change of variables or using the standard formula. To use the standard formula we first need to put it into standard form:
$$u_{tt} - c^2 u_{xx} = F(x,t)$$
We put $y = t$ and take $c = 1/2$:
$$\frac{\partial^2 f_1}{\partial y^2} - \frac{1}{4}\frac{\partial^2 f_1}{\partial x^2} = \frac{x^2 y}{4} - \frac{y}{2} - \frac{1}{2}.$$

The general solution to this equation is
$$f_1(x,y) = p(x+cy) + q(x-cy) + \frac{1}{2c}\int_0^y \int_{x-c(y-y')}^{x+c(y-y')} F(x',y')\,\mathrm{d}x'\,\mathrm{d}y'$$
in which $c = 1/2$ and $F(x,y) = x^2y/4 - y/2 - 1/2$. Substituting these in, the integral becomes
$$\begin{aligned}
& \frac{1}{4}\int_0^y \int_{x-(y-y')/2}^{x+(y-y')/2} (x')^2 y' - 2y' - 2\,\mathrm{d}x'\,\mathrm{d}y' \\
= \quad & \frac{1}{4}\int_0^y \left[(x')^3 y'/3 - 2x'y' - 2x'\right]_{x'=x-(y-y')/2}^{x+(y-y')/2}\,\mathrm{d}y' \\
= \quad & \frac{1}{4}\int_0^y x^2 y'(y-y') + 2(y'-y)(y'+1) + y'(y-y')^3/12\,\mathrm{d}y' \\
= \quad & \frac{1}{2}\int_0^y -y - y'\left[y - 1 - \frac{x^2 y}{2} - \frac{y^3}{24}\right] - (y')^2\left[\frac{y^2}{8} - 1 + \frac{x^2}{2}\right] + \frac{y(y')^3}{8} - \frac{(y')^4}{24}\,\mathrm{d}y' \\
= \quad & \frac{1}{2}\left[-yy' - \frac{(y')^2}{2}\left(y - 1 - \frac{x^2 y}{2} - \frac{y^3}{24}\right) - \frac{(y')^3}{3}\left(\frac{y^2}{8} - 1 + \frac{x^2}{2}\right) + \frac{y(y')^4}{32} - \frac{(y')^5}{120}\right]_{y'=0}^{y} \\
= \quad & \frac{x^2y^3}{24} - \frac{y^2}{4} - \frac{y^3}{12} + \frac{y^5}{960}
\end{aligned}$$
Thus the general solution is
$$f_1(x,y) = p(2x+y) + q(2x-y) + \frac{x^2y^3}{24} - \frac{y^2}{4} - \frac{y^3}{12} + \frac{y^5}{960}.$$

Our boundary conditions $f(x, 0) = 0$ and $f_y(x, 0) = 0$ give

$$0 = p(2x) + q(2x) \qquad 0 = p'(2x) - q'(2x)$$

so $p = q = 0$ and the solution is

$$f_1(x, y) = \frac{x^2 y^3}{24} - \frac{y^2}{4} - \frac{y^3}{12} + \frac{y^5}{960}.$$

Check:

$$f_x = \frac{xy^3}{12} \qquad f_{xx} = \frac{y^3}{12}$$

$$f_y = \frac{x^2 y^2}{8} - \frac{y}{2} - \frac{y^2}{4} + \frac{y^4}{192}.$$

$$f_{yy} = \frac{x^2 y}{4} - \frac{1}{2} - \frac{y}{2} + \frac{y^3}{48}.$$

Thus:

$$f_{xx} - 4f_{yy} = \frac{y^3}{12} - 4\left(\frac{x^2 y}{4} - \frac{1}{2} - \frac{y}{2} + \frac{y^3}{48}\right) = -x^2 y + 2 + 2y$$

as required.

The full solution is

$$f(x, y) = 2y + \frac{\varepsilon}{24}\left(x^2 y^3 - 6y^2 - 2y^3 + \frac{y^5}{40}\right) + O(\varepsilon^2).$$

Carrying out the second calculation via the change of variables $\eta = 2x + y$, $\xi = 2x - y$ gives the same result after rather more algebra.

2. Use the mapping

$$w(z) = i\left(\frac{1-z}{1+z}\right),$$

to find the (exact) solution to Laplace's equation, $\phi_{xx} + \phi_{yy} = 0$, on the unit disc such that $\phi = A$ on the upper half ($y > 0$) of the unit circle $x^2 + y^2 = 1$, and $\phi = B$ on the lower half of the unit circle (where A and B are different constant values). Numerically create a surface plot of your solution in the unit circle for the case $A = 1$ and $B = -1$.

Solution:

$w(z)$ maps the unit disc to the upper half plane. Writing $w = u + iv$ we have

$$u = \frac{2y}{(1+x)^2 + y^2} \quad \text{and} \quad v = \frac{1(x^2+y^2)}{(1+x)^2 + y^2}.$$

In particular, if $x^2 + y^2 = 1$, we have $u = y/(1+x)$ and $v = 0$ and so the image of the unit circle is the real line. The image of the upper half unit circle is the right part of the real line $u > 0$ and the image of the lower half unit circle is $u < 0$. In the w-plane, we need to solve $\phi_{uu} + \phi_{vv} = 0$ in the upper half plane subject to the boundary condition on the real line that $\phi = A$ for $u > 0$ and $\phi = B$ for $u < 0$ and the solution must be bounded at infinity. The unique solution to this problem is

$$\phi = A - (A-B)\frac{\theta}{\pi},$$

where θ is the angular coordinate in the w-plane. This yields

$$\phi = A - \frac{(A-B)}{\pi}\arctan(v/u) = A - \frac{(A-B)}{\pi}\arctan\left(\frac{1-(x^2+y^2)}{2y}\right),$$

as the solution on the unit disc.