Analytical Methods exam question 2022.

1. Consider the following partial differential equation for $f(x, y)$:

$$2\frac{(y+1)}{x}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} - f + \lambda\left(\frac{\partial^2 f}{\partial x^2} - 4\frac{\partial^2 f}{\partial y^2}\right) = -x^2 y$$

in the domain $-\infty < x < \infty$, $y \geq 0$. If $\lambda \gg 1$ and the boundary conditions are given by

$$f(x, 0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x, 0) = 2,$$

determine the solution $f(x, y)$ up to and including terms of order $1/\lambda$.

2. Use the mapping

$$w(z) = i\left(\frac{1-z}{1+z}\right),$$

to find the (exact) solution to Laplace's equation, $\phi_{xx} + \phi_{yy} = 0$, on the unit disc such that $\phi = A$ on the upper half ($y > 0$) of the unit circle $x^2 + y^2 = 1$, and $\phi = B$ on the lower half of the unit circle (where A and B are different constant values). Numerically create a surface plot of your solution in the unit circle for the case $A = 1$ and $B = -1$.