# FOLIATIONS AND THREE-DIMENSIONAL MANIFOLDS

*Mehdi Yazdi*

*This are notes for the course* Foliations and Three-dimensional Manifolds *for the London Taught Course Center.*

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CONTENTS

# 1 BASIC DEFINITIONS AND EXAMPLES

## 1.1 *What is a foliation?*

Given smooth manifolds M and N, a smooth map $f\colon N \to M$ is an embedding if it is a homeomorphism onto its image $f(N)$. An immersion $f\colon N \to M$ is a smooth map whose derivative $Df\colon TM_p \to TN_p$ is injective at every point $p \in N$. In general the image of an immersion can intersect itself, like the figure 8 in the plane which can be thought as an immersion $f\colon S^1 \to \mathbb{R}^2$. An injectively immersed submanifold N of M is the image of an immersion $f\colon N \to M$ such that $f$ is injective. In this case N is allowed to accumulate on itself, but is not allowed to intersect itself.

A two-dimensional *foliation* of a 3-dimensional manifold M is a decomposition of M into injectively immersed surfaces such that locally they form a product $\mathbb{R}^2 \times \mathbb{R}$ by surfaces $\mathbb{R}^2 \times \{\text{point}\}$. The connected components of surfaces in this decomposition are called *leaves*. So, similar to manifolds, foliations are mathematical objects whose local structure is easy to understand while their global structure can be complicated. More formally, a *k-dimensional foliation of an n-manifold* is given by an open covering $\{U_\alpha\}$ of M and homeomorphisms $\phi_\alpha\colon U_\alpha \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that the change of coordinate maps

$$\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1}$$

are of the form

$$\phi_{\alpha\beta}(x, y) = (f(x, y), g(y))$$

where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$, and the cocycle condition

$$\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$$

is satisfied. The $U_\alpha$ are called the *foliation charts*. Note that the second coordinate, $g(y)$, is only a function of $y$. In other words, if the $y$-coordinate of a point $(x, y) \in U_\alpha$ is a fixed constant and $x$ is variable, then the $y$-coordinate of that point viewed from the perspective of an overlapping chart $U_\beta$ is also fixed (by a possibly different constant); hence the sheets $\mathbb{R}^k \times \{\text{point}\}$ patch together to form $k$-dimensional submanifolds of M. The connected components of theses submanifolds are the *leaves* of the foliation. The connected components of the leaves in a foliated chart are called *plaques*; these are the sheets $\mathbb{R}^k \times$ point. In this note, we always assume the function $f$ to be $C^\infty$. If the function $g$ is $C^r$ for $r = 0, 1, 2, \cdots, \infty$, then the foliation is considered to be of smooth class $C^r$. In other words, the leaves are smoothly injectively immersed but the change of coordinate maps can be only $C^r$ in the transverse direction.

A foliation is *orientable* if there is a consistent choice of orientation for its leaves. A foliation is *transversely orientable* (or *co-orientable*) if there is a consistent choice of transverse orientation on its leaves. If the ambient manifold is orientable, then a foliation is orientable if and only if it is transversely

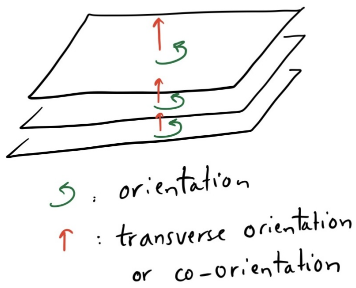


*Figure 1: Orientation and transverse orientation of a foliation in a chart*

orientable. See Figure 1.

1.1 EXAMPLE. Let $f: S^1 \to S^1$ be a homeomorphism. Let $M_f$ be the mapping torus of $f$, that is $M_f := S^1 \times [0,1]/\sim$ where $(x, 1) \sim (f(x), 0)$ for every $x \in S^1$. Then $M_f$ is homeomorphic to either a torus or a Klein bottle, according to whether $f$ is orientation-preserving or not. The product foliation on $S^1 \times [0,1]$ by leaves $\{\text{point}\} \times [0,1]$ induces a one-dimensional foliation on $M_f$. If $f$ is a rotation by angle $\theta$, then we obtain a linear foliation of slope $\theta$ on the torus $\mathbb{T}^2$. Note that if $\theta$ is irrational, then every leaf is dense; the global behaviour of leaves can be complicated.

Given any equivalence relation $\sim$ on a topological space X, there is a quotient topology on the set $X/\sim$ of equivalence classes, where a set $U \subset X/\sim$ is open if and only if its preimage under the natural map $\pi: X \to X/\sim$ is an open subset of X. Given a group G acting on a topological space X, we can endow the quotient space $X/G$ with the quotient topology. Note that the points in $X/G$ correspond to the orbits of the action of G on X. The next proposition gives a general way of obtaining foliations as quotients.

1.1 PROPOSITION. Let M be a, possibly non-compact, manifold and $\mathcal{F}$ be a foliation on M. Let G be a group acting on M by homeomorphisms such that the action of G is a covering action and that locally G sends leaves to leaves; i.e. for every $x \in M$ there is a open neighbourhood U of $x$ foliated as a product $\mathbb{R}^k \times \mathbb{R}^{n-k}$ such that for every $g \in G$, $gU$ is an open neighbourhood of $gx$ foliated as a product $\mathbb{R}^k \times \mathbb{R}^{n-k}$ with $g: U \to gU$ sending leaves to leaves and such that all the neighbourhoods $gU$ for $g \in G$ are pairwise disjoint. Then $\mathcal{F}$ induces a foliation on the quotient manifold $M/G$.

*Proof.* □

1.2 EXAMPLE (Reeb foliation). Consider the graph of the function $f: (-1, 1) \to \mathbb{R}$ defined as

$$f(x) = -\log(1 - x^2).$$

Note that $\lim_{x \to \pm 1} f(x) = +\infty$ and the translates of the graph of $y = f(x)$ in

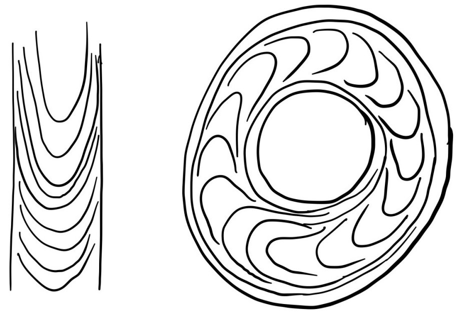

*Figure 2: Right: 2-dimensional Reeb component*

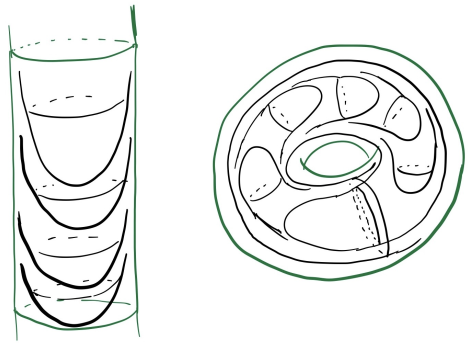

*Figure 3: Right: 3-dimensional Reeb component*

the $y$-direction give a foliation of the open infinite cylinder $(-1,1)\times\mathbb{R}$. This can be extended to a foliation of the closed infinite cylinder $M = [-1,1]\times\mathbb{R}$ by adding the boundary leaves $\{-1,1\}\times\mathbb{R}$. There is a free and discrete action of the group $\mathbb{Z}$ on M, the vertical translation by 1 unit, that sends leaves to leaves. The induced foliation on the quotient manifold $M/G = [-1,1]\times S^1$ is by definition the *Reeb foliation* in dimension two. See Figure 2. By using the graph of the function $f\colon \mathrm{int}(\mathbb{D}^n)\to\mathbb{R}$ defined as $f(x) = -\log(1-r^2)$ instead ($r$ = radial distance to the origin in $\mathrm{int}(\mathbb{D}^n)$), we can similarly define the Reeb foliation of $\mathbb{D}^n\times S^1$ in dimension $n$. See Figure 3 for $n=3$. Note that every leaf of the Reeb foliation in the interior of the manifold $\mathbb{D}^n\times S^1$ is topologically an $n$-dimensional plane, and limits to the boundary leaf $\partial\mathbb{D}^n\times S^1$.

The 3-sphere $\mathbb{S}^3 = \mathbb{R}^3\cup\{\infty\}$ can be written as a union of two solid tori glued together along their common boundary. See Figure 4. By putting a Reeb foliation on each solid torus, Reeb gave the first example of a two-dimensional foliation of $\mathbb{S}^3$.

1.3 EXAMPLE (Poincaré–Hopf index formula). The only closed surfaces that admit a codimension-one foliation are those with Euler characteristic 0, namely the torus and the Klein bottle. This follows from the *Poincaré–Hopf index formula*: given a smooth vector field F with isolated zeros (also called singularities) on a closed manifold M, sum of the indices of the singularities of F is

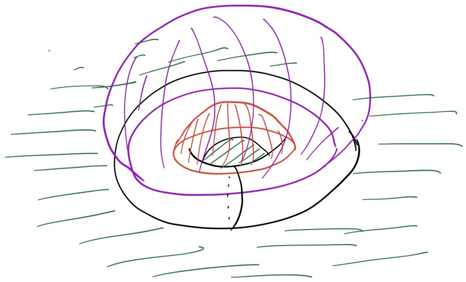

*Figure 4: The complement of the standard solid torus in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ is another solid torus. If we think of a solid torus as a disjoint union of discs parametrised by $S^1$, then 4 of these discs are shown in this figure. Note that the union of the horizontal annulus and the point $\infty$ is one such disc. One can think of the parametrising $S^1$ as the union of $\infty$ and the vertical line through the origin.*

equal to the Euler characteristic of M. In particular, this sum does not depend on the choice of the vector field and is a topological invariant of M. If M has boundary, the same result holds if the vector field F is pointing outward along all boundary components of M. See Milnor [MW97][Chapter 6].

Now, let $\mathcal{F}$ be a codimension-one foliation on M. Assume that $\mathcal{F}$ is transversely orientable for the moment. Fix a Riemannian metric M, and let F be the oriented unit normal vector to $\mathcal{F}$. Then F has no singularity, and so by Poincaré–Hopf formula we must have $\chi(M) = 0$. If $\mathcal{F}$ is not transversely orientable, then there is a double cover $\tilde{M} \to M$ where the lift of $\mathcal{F}$ to $\tilde{M}$ becomes transversely orientable. Since $\chi(\tilde{M}) = 2\chi(M)$, the result follows.

1.4 EXAMPLE (Foliations from submersions). Let $f\colon M \to N$ be a smooth submersion between manifolds, without boundary, of dimensions respectively $n$ and $k$; i.e. the derivative $Df\colon TM \to TN$ is surjective. By the submersion theorem, for each point $m \in M$ there are neighbourhoods U of $m$ parametrised by $(x_1, \cdots, x_n)$, and V of $f(m)$ parametrised by $x_1, \cdots, x_q$ relative to which the map $f$ has the form of the projection

$$f(x_1, \cdots, x_n) = (x_1, \cdots, x_k).$$

Hence, the level sets of $f$ locally form a product $\mathbb{R}^{n-k} \times \mathbb{R}^k$, giving rise to a codimension-$k$ foliation of M.

1.5 EXAMPLE (Example 1.1.2 of [CC00a]). Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = (x^2 - 1)e^y.$$

We have $\frac{\partial f}{\partial x} = 2xe^y$ and $\frac{\partial f}{\partial y} = (x^2 - 1)e^y$. Therefore the equation $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (0, 0)$ has no solution and so $f$ is a submersion. The level sets of $f$ are as in Figure 5.

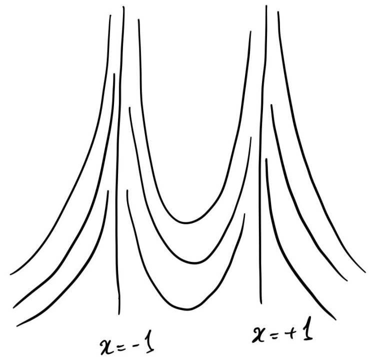


*Figure 5: Example of a foliation defined by level sets of a submersion from $\mathbb{R}^2$ to $\mathbb{R}$*

## 1.2 *Holonomy*

The exposition in this section follows Camacho and Neto [CN13].

Let $\mathcal{F}$ be a codimension-$k$ foliation of a manifold M and $\gamma : [0,1] \to \mathrm{M}$ be a path lying in a leaf L of $\mathcal{F}$. We allow the possibility that $\gamma$ is closed. Let $\mathrm{D}_a$ and $\mathrm{D}_b$ be transversals around $\gamma(0) = a$ and $\gamma(1) = b$. Intuitively we can flow the points of $\mathrm{D}_a$ in a neighbourhood of $a$ along the leaves of the foliation above $\gamma$ and map them to points of $\mathrm{D}_b$ in a neighbourhood of $b$. We will make this idea precise, and furthermore show that the resulting map, called holonomy along $\gamma$, only depends on the homotopy class of the path $\gamma$ in L relative to its endpoints. Note that this is a generalisation of Poincaré's first return map.

Given any point $p$ on $\gamma$, by looking at a foliation chart around $p$, it is easy to see that such a holonomy map can be defined for a small open neighbourhood $\mathrm{U}_p$ of $p$ in $\gamma$. Clearly $\{\mathrm{U}_p\}$ is a cover of $\gamma$, and by compactness it admits a finite subcover. Hence, there is a sequence $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$ and local foliation charts $\mathrm{U}_i$ such that $\gamma([t_i, t_{i+1}])$ lies in $\mathrm{U}_i$. Choose transverse $k$-dimensional discs $\mathrm{D}_i$ based as $\gamma(t_i)$, where $\mathrm{D}_0 = \mathrm{D}_a$ and $\mathrm{D}_{m+1} = \mathrm{D}_b$. For each $i$, the plaque of $\mathrm{U}_i$ intersecting $\mathrm{D}_i$ in a small neighbourhood of $\gamma(t_i)$ intersects $\mathrm{D}_{i+1}$ in a unique point, and hence we have a well-defined map $f_i$ from a small neighbourhood of $\mathrm{D}_i$ to a small neighbourhood of $\mathrm{D}_{i+1}$. Note that $f_i$ is an injective continuous map between manifolds (small transverse discs) of the same dimension, and hence $f_i$ is a $\mathrm{C}^r$ diffeomorphism if $\mathcal{F}$ is $\mathrm{C}^r$. Define $f$ as the composition

$$f = f_m \circ \cdots \circ f_1 \circ f_0,$$

which is defined on a possibly smaller neighbourhood of $\gamma(0) = 0$. This map $f$ is called the *holonomy* of $\mathcal{F}$ along $\gamma$.

1.6 DEFINITION. Given two functions $f, g \colon \mathrm{X} \to \mathrm{Y}$ between topological spaces X and Y and a point $x \in \mathrm{X}$, we say that $f$ and $g$ have the *same germ at $x$* if there is an open neighbourhood U of $x$ such that the restrictions of $f$ and $g$ to U coincide. This defines an equivalence relation on the set of functions from X to Y. The *germ of $f$ at $x$* is defined as its equivalence class.

1.2 Lemma. 1. Fixing the endpoint transversals $D_a$ and $D_b$, the germ of the map $f$ at $\gamma(0) = a \in D_a$ does not depend on the choice of the intermediate points $t_i \in (0,1)$ and the transverse discs $D_i$.

2. If $\gamma$ and $\gamma'$ are homotopic relative to their endpoints in L, then the holonomy maps along $\gamma$ and $\gamma'$ are equal.

1.7 proposition. *Let $\mathcal{F}$ be a $C^r$ codimension-$k$ foliation of a manifold* M. *Let* $p \in M$ *be a base point lying on a leaf* L, *and* D *be a transverse $k$-dimensional disc around $p$. The holonomy map defines a homomorphism*

$$\mathrm{hol}\colon \pi_1(L, p) \to G(D, p),$$

*where* $G(D, p)$ *is the group of germs of* $C^r$ *diffeomorphisms of* D *at $p$.*

1.8 example. Consider the Reeb foliation of $\mathbb{S}^3$. Let L be the torus leaf and T be one of the two solid tori bounded by L. Pick $\gamma \in \pi_1(L)$ such that $\gamma$ bounds a disc in T; i.e. $\gamma$ is a *meridian* of T. Then the holonomy of $\gamma$ on the side of L contained in T is the germ of the identity homeomorphism of $[0,1]$. Moreover, the holonomy of $\gamma$ on the side of L not contained in T is the germ of a *shift* map; i.e. a homeomorphism $f\colon [0,1] \to [0,1]$ such that $f(0) = 0$ and either

- $f(x) < x$ for every $x \in (0,1)$; or
- $f(x) > x$ for every $x \in (0,1)$.

The next theorem states that the germ of a foliation in a neighbourhood of a *compact* leaf L is completely determined by the holonomy along L. To state this result, we need some preliminary setup. Note that the holonomy map for a leaf depends on the choice of a transversal as well. However, varying the transversal changes the holonomy map only by a conjugation (A conjugate of $f$ is a map of the form $g \circ f \circ g^{-1}$ where $g$ is a homeomorphism. It is useful to think of a conjugation as 'renaming' the points of the ambient space via the map $g$). Let L and L′ be compact leaves of $C^r$ codimension-$k$ foliations $\mathcal{F}$ and $\mathcal{F}'$ respectively. The holonomies of L and L′ are $C^r$-*conjugate* when there exists

- transversals D and D′ to L and L′;
- basepoints $p \in D \cap L$ and $p' \in D' \cap L'$;
- a homeomorphism $f\colon L \cup D \to L' \cup D'$ such that $f(p) = p'$, and $f|D$ and $f|L$ are $C^r$-diffeomorphisms, and for every $[\gamma] \in \pi_1(L, p)$ and every $x'$ sufficiently close to $p'$

$$f \circ \mathrm{hol}_\gamma \circ f^{-1}(x') = \mathrm{hol}_{f\circ\gamma}(x').$$

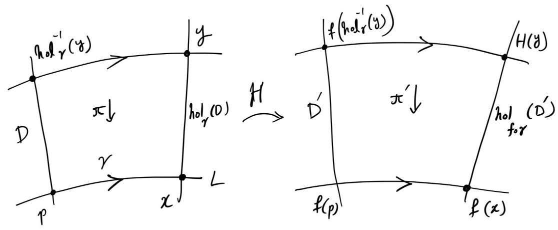


*Figure 6: Defining the conjugating map* H

1.3 THEOREM. Let L and L′ be compact leaves of $C^r$ codimension-$k$ foliations $\mathcal{F}$ and $\mathcal{F}'$. Then the holonomy maps of L and L′ are $C^r$-conjugate if and only if there are neighbourhoods $N \supset L$ and $N' \supset L'$ and a $C^r$-diffeomorphism $H\colon N \to N'$, with $H(L) = L'$ and with H taking leaves of $\mathcal{F}|N$ to leaves of $\mathcal{F}'|N'$. In this case we say that $\mathcal{F}$ and $\mathcal{F}'$ are locally equivalent on L and L′.

*Proof.* We only sketch the proof and leave some of the details as an exercise. Let $\mathcal{F}$ and $\mathcal{F}'$ be locally equivalent on L and L′ via a $C^r$-diffeomorphism $H\colon N \to N'$, then it is easy to see that the holonomies of L and L′ are $C^r$-conjugate. For example pick a transversal D for L and a basepoint $p \in D \cap L$ and define $p' = H(p)$ and $D' = H(D)$. Given $\gamma \in \pi_1(L, p)$, a chain of charts for $\mathcal{F}$ used to define $\mathrm{hol}_\gamma$ is mapped by H to a chain of charts for $\mathcal{F}'$. It follows that the holonomies of L and L′ are $C^r$-conjugate.

Now assume that the holonomies of L and L′ are $C^r$-conjugate. We use the following claim, whose proof we leave as an exercise: there is a neighbourhood N of the compact leaf L and a retraction $\pi\colon N \to L$ such that for each $x \in L$ the fiber $\pi^{-1}(x)$ is transverse to $\mathcal{F}$.

Note that we do not assume that the neighbourhood N can be chosen to be a *saturated* neighbourhood; i.e. a union of leaves of $\mathcal{F}$. In particular, the leaves of $\mathcal{F}$ might exit any such N; for example think of L being the torus leaf of a Reeb foliation.

Choose such retractions $\pi\colon N \to L$ and $\pi'\colon N' \to L'$. Let $f\colon L \cup D \to L' \cup D'$ be a homeomorphism that conjugates the holonomies of L and L′. The retractions can be chosen such that $D = \pi^{-1}(p)$ and $D' = \pi'^{-1}(f(p))$. Given a point $x \in L$, choose a path $\gamma$ from $p$ to $x$. Define $H(y)$ for $y \in \mathrm{hol}_\gamma(D)$ as

$$H(y) = \mathrm{hol}_{f(\gamma)}(f(\mathrm{hol}_\gamma^{-1}(y))).$$

See Figure 6. Note that every point in N lies one a transverse discs $\mathrm{hol}_\gamma(D)$ for some $\gamma$, so the above definition covers all points of N. One can check that this definition is independent of the choice of $\gamma$. The defined map H is a $C^r$-diffeomorphism.

□

1.9 EXERCISE. Prove the claim in the proof of Theorem 1.3.

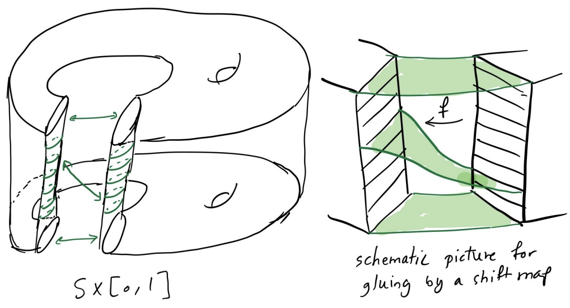


*Figure 7: Gluing operation*

We will see later that in some special cases (such as in suspension foliations) it is possible to define holonomy as global maps rather than germs of maps.

## 1.3 *Operations for modifying foliations*

1.4 Operation (Cutting). Let $\mathcal{F}$ be a foliation of a manifold M and $S \subset M$ be an embedded codimension-one submanifold that is either transverse to $\mathcal{F}$ or is a leaf of $\mathcal{F}$. Then we can cut the manifold M along S and obtain an induced foliation on it.

1.5 Operation (Gluing). Let $M_1$ and $M_2$ be two foliated manifolds with boundary. Let $T_1$ and $T_2$ be boundary components of $M_1$ and $M_2$ such that the induced foliations on them are smoothly conjugate; i.e. there is a diffeomorphism $f\colon T_1 \to T_2$ sending leaves to leaves. Then we can glue $M_1$ and $M_2$ to obtain a foliation of $M_1 \cup_f M_2$. Similarly if M is a foliated manifold and $T_1$ and $T_2$ are boundary components of M whose induced foliations are smoothly conjugate, then we can glue M to itself by gluing $T_1$ and $T_2$ and obtain a new foliation.

1.10 example. Let S be a compact surface, $C_1$ and $C_2$ be two distinct boundary components of S, and $\mathcal{F}$ be the product foliation on $S \times [0,1]$ by leaves $S \times \{\text{point}\}$. Let $f\colon [0,1] \to [0,1]$ be a homeomorphism such that $f(0) = 1$, $f(1) = 1$, and $f(x) > x$ for every $x \in (0,1)$. Define a homeomorphism

$$\hat{f}\colon C_1 \times [0,1] \to C_2 \times [0,1],$$

$$\hat{f}(t,x) = (t, f(x)).$$

Glue the foliated manifold $(S \times [0,1], \mathcal{F})$ to itself using the map $\hat{f}$ to obtain a new foliation $\mathcal{F}'$ on $M'$. Then $M'$ is diffeomorphic to $\hat{S} \times [0,1]$, where $\hat{S}$ is obtained from S by gluing the boundary components $C_1$ and $C_2$. Each leaf of $\mathcal{F}'$ in the interior of $M'$ is non-compact and limits to the leaves $\hat{S} \times \{0,1\}$. See Figure 7.

1.11 example (Sheer). Let $\mathcal{F}$ be a codimension-one foliation of a 3-manifold M.

Let $\alpha$ be a simple closed curve in a leaf L of $\mathcal{F}$ such that there is a *fence* above $\alpha$ that is foliated as a product. By this we mean that there is an embedding $\alpha \times [-1, 1] \hookrightarrow M$ with $\alpha \times 0 = \alpha$ such that $\alpha \times [-1, 1]$ is transverse to $\mathcal{F}$, and the induced foliation on $\alpha \times [0, 1]$ is the product foliation. Then we can cut the foliation along $\alpha \times [0, 1]$ and reglue the two copies of $\alpha \times [0, 1]$ together via an orientation-preserving homeomorphism of $[0, 1]$. Starting from the product foliation of $M = S \times S^1$, by repeatedly applying sheers, one can obtain complicated foliations of $M = S \times S^1$.

1.12 REMARK. Given a manifold M and an injectively immersed codimension-one submanifold L of M, cutting M along L, denoted by $M \setminus \setminus L$, is the metric completion of $M - L$ with respect to the path metric. When L is two-sided (i.e. transversely orientable), $M \setminus \setminus L$ is obtained from $M - L$ by adding two copies of L.

1.6 OPERATION (Turbulisation). Let $\mathcal{F}$ be a codimension-one foliation of a 3-manifold such that $\mathcal{F}$ is both orientable and transversely orientable. Let $\gamma$ be a closed transversal for $\mathcal{F}$, and $N(\gamma)$ be a regular neighbourhood of $\gamma$ diffeomorphic to a solid torus $\mathbb{D}^2 \times S^1$ with the product foliation by discs. The induced foliation on the torus $\partial N(\gamma)$ is the product foliation $S^1 \times S^1$. Remove the foliation in the interior of the solid torus. Then 'Spin' the leaves of $\mathcal{F}|\partial N(\gamma)$ to converge to a torus $T \subset N(\gamma)$ parallel to $\partial N(\gamma)$. Then T bounds a solid torus contained in $N(\gamma)$, and we can insert a Reeb foliation inside this solid torus. The resulting foliation is called a turbulisation of $\mathcal{F}$ along the closed transversal $\gamma$. The turbulisation operation can similarly be done in other dimensions as well.

Note that the Reeb foliation of a solid torus can be obtained by starting from the product foliation of a solid torus, spinning the leaves intersecting the boundary, and finally adding the limiting two-dimensional torus leaf.

We need the notion of a *Dehn twist* for the next operation. Let $A = S^1 \times [0, 1]$ be an annulus, where $S^1 = \mathbb{R}/\mathbb{Z}$. Define a homeomorphism of A fixing $\partial A$ pointwise as follows:

$$f(\theta, x) = (\theta + x, x).$$

See Figure 8.

1.7 OPERATION (Spinning). Let $\mathcal{F}$ be a codimension-one transversely oriented foliation of an orientable 3-manifold, and T be a torus component of $\partial M$. Assume that the foliation $\mathcal{F}|T$ is the mapping torus of an orientation-preserving homeomorphism $f : S^1 \to S^1$. Hence $T = S^1 \times S^1$ and $\mathcal{F}|T$ is transverse to the circles $S^1 \times$ point. We describe how to *spin* the leaves of $\mathcal{F}$ in a neighbourhood of T so that T becomes a leaf of the new foliation. Let $N_1 = T \times [0, 1]$ be a neighbourhood of T with $T \times 0 = T$. Let $f_1$ be a homeomorphism of $N_1 = T \times [0, 1]$ such that the restriction of $f_1$ to each annulus $S^1 \times \text{point} \times [0, 1]$ is a Dehn twist. We can think of $f_1$ as a homeomorphism of M by extending $f_1$ via identity to

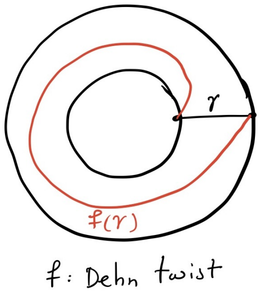


*Figure 8: The image of an arc $\gamma$ under the Dehn twist is shown.*

$M - N_1$. Note that $f_1$ spins the leaves of $\mathcal{F}$ once around $S^1 \times$ point direction in T, which is the direction transverse to the leaves of $\mathcal{F}|T$. Similarly, define homeomorphisms $f_n$ of $N_n = T \times [0, \frac{1}{n}]$ and extend them by identity to the rest of M. Let $f$ be the infinite composition $\cdots \circ f_3 \circ f_2 \circ f_1$. Then $f$ is well-defined since

- The restriction of each $f_i$, and hence $f$, to T is the identity map.
- Define the support of a self-homeomorphism $h$ as the set of points $x$ in its domain such that $f(x) \neq x$; i.e. points that are moved by $h$. Then each point $p$ in $M - T$ lies in the support of at most finitely many $f_i$, and so $f(p)$ is well-defined.

We say that the foliation $f(\mathcal{F})$ is obtained by spinning the leaves of $\mathcal{F}$ around T. Note that the result of the operation also depends on the choice of the product structure $S^1 \times S^1$.

1.13 REMARK. Even if $\mathcal{F}$ is transversely smooth, the foliation obtained by spinning might not be transversely smooth along T. Such examples can be constructed using *Kopell's Lemma* concerning commuting diffeomorphisms of $[0, 1]$.

1.8 OPERATION (Denjoy blow-up). Let $f$ be an orientation-preserving homeomorphism of $S^1$, and $\mathcal{F}$ be the induced foliation on the mapping torus of $f$. Fix a point $x_0$ on $S^1$, and let $x_n := f^n(x)$ be the orbit of $x$ for $n \in \mathbb{Z}$. Replace each point $x_n$ by an interval $I_n$ such that the total length of the intervals $I_n$ is bounded. This is called *blowing up the orbit of* $x_0$. Hence after blowing up the orbit of $x_0$, the circle $S^1$ is replaced by a new circle C. For each $n \in \mathbb{Z}$ choose an orientation-preserving homeomorphism $h_n \colon I_n \to I_{n+1}$. The map $f$ together with the collection of homeomorphisms $\{h_n\}_{n\in\mathbb{Z}}$ induce a foliation $\mathcal{F}'$ of $C \times S^1$ via the mapping torus construction. See Figure 10. We say that $\mathcal{F}'$ is obtained from $\mathcal{F}$ by a *generalised Denjoy blow-up*. Geometrically, we blow-up a leaf L of $\mathcal{F}$ (corresponding to the orbit of $x_0$) and replace it with $L \times [0, 1]$. Inside $L \times [0, 1]$ we insert a foliation transverse to $[0, 1]$ factor. If L is a line then any such foliation of $L \times [0, 1]$ is isotopic to a product foliation by leaves $L \times$ point,

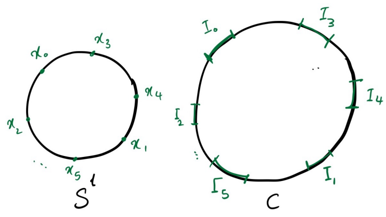


*Figure 9: Blowing up an orbit*

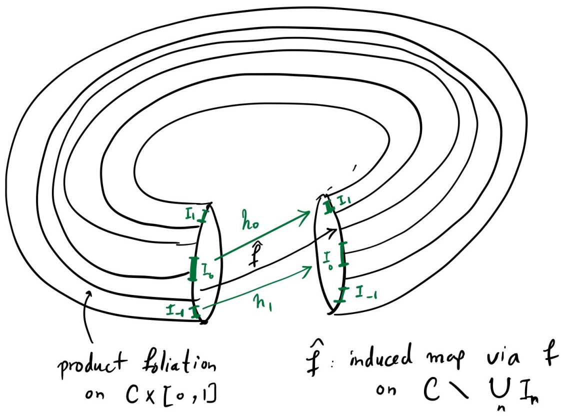


*Figure 10: Generalised Denjoy blow-up*

but if L is a circle, then we can insert any foliation of $L \times [0,1]$ obtained from an orientation-preserving homeomorphism of $[0,1]$ via the mapping torus construction.

1.14 REMARK. Even if $\mathcal{F}$ is smooth, a Denjoy blow-up of $\mathcal{F}$ might not be smooth. Denjoy used this operation to construct homeomorphisms of $S^1$ that have irrational *rotation number* but are not topologically conjugate to a rotation, answering a question of Poincaré.

## 1.4 *Suspension foliations*

This section mainly follows Camacho and Neto [CN13]. Intuitively, a fiber bundle is a family of manifolds homeomorphic to a fixed manifold F, called the fiber, and parametrised by the points in another manifold B called the base. More precisely, a *fiber bundle* consists of differentiable manifolds E, B, and F and a differentiable map $\pi\colon E \to B$ such that there is a cover $\{U_i\}$ of B and diffeomorphisms $\phi_i\colon \pi^{-1}(U_i) \to U_i \times F$ such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\phi_i"] \arrow[dr, "\pi"'] & U_i \times F \arrow[d, "p"] \\ & U_i \end{tikzcd}$$

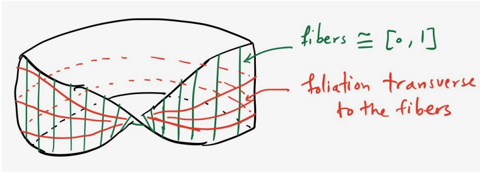


*Figure 11: The Möbius band has the structure of a* $[0, 1]$*-bundle over* $S^1$*. A leaf of a foliation transverse to the fibers is shown.*

where $p$ is the projection onto the first factor. In particular $\pi$ is a submersion. The manifold E is the *total space*, F is the *fiber*, and B is the *base* of the fiber bundle.

A continuous map $p\colon Y \to X$ is a covering map if there is a discrete space D such that every $x \in X$ has a neighbourhood $U_x$ where $p^{-1}(U_x)$ is a disjoint union of neighbourhoods $V_d$ for $d \in D$ such that each $p|V_d\colon V_d \to U_x$ is a homeomorphism. The degree of a covering map is the cardinality of D. A covering map is called universal covering if Y is simply connected. In this case $\pi_1(X)$ acts on the universal cover Y via deck transformations (shuffling the points in $p^{-1}$(point) as cards in a deck). If X is connected and locally simply connected then the universal covering exists. Covering maps are fiber bundles whose fibers are a discrete set of D points.

Let $\pi\colon E \to B$ be the projection map of a fiber bundle with total space E, fiber F, and base B. A foliation $\mathcal{F}$ on E is *transverse to the fibers* if at each point $x \in E$ the leaf L of $\mathcal{F}$ passing through $x$ is transverse to the fiber above $\pi(x)$ and of complementary dimension, and the projection $p|L\colon L \to B$ is a covering map. The pair $(\pi\colon E \to B, \mathcal{F})$ is called a *foliated bundle*.

1.15 EXAMPLE. The simplest examples of fiber bundles are products $E = B \times F$ with the map $\pi\colon B \times F \to B$ being the projection onto the first factor. Such fiber bundles are sometimes referred to as trivial fiber bundles. The foliation of E by leaves $B \times$ point is transverse to the fibers.

1.16 EXAMPLE. Let $f\colon M \to M$ be a homeomorphism of a manifold M, and $M_f$ be the mapping torus of $f$. Then $M_f$ is the total space of a fiber bundle with fiber M and base $S^1$. If $M = S^1$ then $M_f$ is a torus or a Klein bottle, and if $M = [0, 1]$ then $M_f$ is an annulus or a Möbius band. The foliation by points on M induces, via the suspension construction, a one-dimensional foliation $\mathcal{F}$ of $M_f$ that is transverse to the fibers. See Figure 11.

1.17 EXAMPLE. Let S be a surface with two distinguished boundary components $b_1$ and $b_2$. Consider $S \times [0, 1]$ with the product foliation. The induced foliations on the annuli $b_i \times [0, 1]$ are products. Hence we can glue the annuli $b_i \times [0, 1]$ together according to an orientation-preserving homeomorphism $f\colon [0, 1] \to [0, 1]$ to obtain a foliation of $\hat{S} \times [0, 1]$ transverse to $[0, 1]$ fibers, where $\hat{S}$ is obtained from S by gluing the boundary components $b_1$ and $b_2$ together.

1.18 EXAMPLE. Let $f$ and $g$ be two orientation-preserving homeomorphisms of $[0,1]$ that commute with each other; i.e. $f \circ g = g \circ f$. For example $f$ and $g$ could be the following:

- $f = h^n$ and $g = h^m$ for some homeomorphism $h$ of $[0,1]$ and $m, n \in \mathbb{Z}$.
- If X is a vector field on $[0,1]$ vanishing at 0 and 1, then $f$ is the time-$t_1$ of the flow of X and $g$ is the time-$t_2$ of the flow of X for fixed $t_1$ and $t_2$.

Start with the product foliation of $\mathrm{I}^2 \times [0,1]$ with leaves $\mathrm{I}^2 \times$ point, and glue a pair of oppose sides according to the map $f$ and the other pair of opposite sides according to the map $g$. More precisely, define an equivalence relation on the boundary of $\mathrm{I}^2 \times [0,1]$ where $(x, y, t) \sim (x', y', t')$ if

- $x = x'$, $y = 0$, $y' = 1$, and $t' = f(t)$; or
- $x = 0$, $x' = 1$, $y = y'$, and $t' = g(t)$.

We check that this equivalence relation induces a foliation on the quotient space. We examine that every point in the quotient space has a neighbourhood with a product structure. This is clear for points in the interior of $\mathrm{I}^2 \times [0,1]$ since they are not identified with any other point. Given a point $(x, y, t)$ on the boundary of $\mathrm{I}^2 \times [0,1]$ and not lying in $(\partial \mathrm{I})^2 \times [0,1]$, it is identified with exactly one other point and so their local (half plane $\times\, \mathbb{R}$) foliated neighbourhoods patch together to give a foliated neighbourhood $\mathbb{R}^2 \times \mathbb{R}$ in the quotient. It remains to inspect the points in $(\partial \mathrm{I})^2 \times [0,1]$. In this case $x, y \in \{0,1\}$. The orbit of $(0,0,t)$ consists of

$$(0,0,t), \quad (0,1,f(t)), \quad (1,0,g(t)), \quad (1,1,f \circ g(t)).$$

Here we use the relation $f \circ g = g \circ f$. Therefore the (quarter plane$\times\mathbb{R}$) foliated neighbourhoods of the 4 points in the equivalence class patch together to give a foliated neighbourhood in the quotient space. This gives a foliation transverse to the fibers of a fiber bundle with fiber $[0,1]$ and with base the torus $\mathbb{T}^2$. Note that the data of $\{f, g\}$ satisfying the relation $f \circ g = g \circ f$ can be rephrased as a homomorphism

$$\pi_1(\mathbb{T}^2) \to \mathrm{Homeo}^+([0,1]).$$

Let $p\colon \mathrm{E} \to \mathrm{B}$ be the projection map of a fiber bundle with fiber F, and $\mathcal{F}$ be a foliation transverse to the fibers of E. Then we can define a *holonomy map*

$$\mathrm{hol}\colon \pi_1(\mathrm{B}, b) \to \mathrm{Diffeo}(\mathrm{F})$$

as follows. Let $\gamma\colon [0,1] \to \mathrm{B}$ with $\gamma(0) = \gamma(1) = b$ be a loop based at $b \in \mathrm{B}$. Identify $p^{-1}(b)$ with F. Let $f \in \mathrm{F} = \pi^{-1}(b)$ be a point and $\mathrm{L}_f$ be the leaf of $\mathcal{F}$ passing through $f$. Since $p|\mathrm{L}_f\colon \mathrm{L}_f \to \mathrm{B}$ is a covering map by definition, we can lift the path $\gamma\colon [0,1] \to \mathrm{B}$ to a unique path $\tilde{\gamma}\colon [0,1] \to \mathrm{L}_f$ with $\tilde{\gamma}(0) = f$. Define the map $\phi_\gamma\colon \mathrm{F} \to \mathrm{F}$ by $\phi_\gamma(f) = \tilde{\gamma}(1)$. Note that $\phi_\gamma$ has an inverse, which is $\phi_{\gamma^{-1}}$. Hence $\phi_\gamma$ is a diffeomorphism of F. Moreover, it only depends on the based homotopy class of $\gamma$, and it defines a homomorphism $\mathrm{hol}\colon \pi_1(B, b) \to \mathrm{Diffeo}(F)$. Note that the holonomy of a foliated fiber bundle is defined globally unlike the holonomy of a leaf of a foliation which is defined as a germ of a local diffeomorphism.

1.9 Theorem (Suspension of a representation). Let B and F be connected manifolds. Given a representation

$$h\colon \pi_1(B) \to \mathrm{Homeo}(F)$$

there is a fiber bundle with fiber F, base B, and total space E, and a foliation $\mathcal{F}$ of E transverse to the fibers whose holonomy is $h$.

*Proof.* Let $p\colon \tilde{B} \to B$ be the projection map of the universal cover $\tilde{B}$ of B, and denote the deck action of $\gamma \in \pi_1(B)$ on $x \in \tilde{B}$ by $\gamma \cdot x$. Consider the product fiber bundle projection $\tilde{B} \times F \to \tilde{B}$ and the foliation $\mathcal{G}$ transverse to the fibers whose leaves consist of $\tilde{B} \times$ point. There is an action of $\pi_1(B)$ on the total space $\tilde{B} \times F$, with $\gamma \in \pi_1(B)$ acting as

$$\gamma \cdot (x, f) = (\gamma \cdot x, h(\gamma)(f)),$$

where $\gamma \cdot x$ in the first factor is the deck action. This action is a covering action, and sends leaves of $\mathcal{G}$ to leaves of $\mathcal{G}$. Hence there is an induced foliation $\mathcal{F}$ on the quotient

$$E := (\tilde{B} \times F)/\pi_1(B).$$

Then E is the total space of a fiber bundle with fiber F and base B, and with projection map $\pi(x, f) = p(x)$ where $(x, f) \in \tilde{B} \times F$. Note that the projection map is well-defined since $p(x) = p(\gamma \cdot x)$ for every $\gamma \in \pi_1(B)$. Moreover $\mathcal{F}$ is a foliation transverse to fibers of E, and the total holonomy of $\mathcal{F}$ is $h$. □

1.19 remark. Note that the topology of the fiber bundle depends on the representation.

1.20 example (I-budnle replacement). We revisit the Denjoy blow-up. Let L be a leaf of a codimension-one transversely orientable foliation $\mathcal{F}$. Replace L by a $[0, 1]$-bundle over L. Insert any transversely orientable foliation of $[0, 1]$-bundle over L transverse to $[0, 1]$ factor. The resulting foliation is called an I-*bundle replacement over* L. The special case where the inserted foliation in $L \times [0, 1]$ is the product foliation is called the *Denjoy blow-up of* L. More generally if L is not transversely orientable then L is replaced by a non-trivial $[0, 1]$-bundle over L and a foliation transverse to the fibers.

## 1.5 *Exercises*

### Exercise 1

Let M be a compact 3-manifold whose boundary is a union of a torus and a genus two surface; for example one such M could be obtained by removing a regular neighbourhood of the union of a knot and a $\Theta$-shape curve from $S^3$. Does M admit a codimension-one transversely orientable foliation tangential to $\partial M$ (i.e. having $\partial M$ as a union of leaves)?

You may assume the following version of the Poincaré–Hopf formula: Let M be a compact 3-manifold with boundary. Let F be a smooth vector field on M with isolated singularities and transverse to $\partial M$. Then sum of the Euler characteristics for the inward pointing components of $\partial M$ (with respect to F) is equal to that of the outward pointing components of $\partial M$.

### Exercise 2

(Exercise 1.1.13 of [CC00a])

a) Find a submersion $f\colon \mathrm{int}(\mathbb{D}^n \times S^1) \to S^1$ whose level sets give the restriction of the Reeb foliation to the interior of $\mathbb{D}^n \times S^1$.

b) Show that there are no submersion from $\mathbb{D}^n \times S^1$ onto a 1-manifold whose level sets give the Reeb foliation (*Hint*: Show that if a level set $f^{-1}(a)$ of a submersion is compact, then every level set that comes sufficiently close to $f^{-1}(a)$ is also compact.)

### Exercise 3

Show that the foliation of $\mathbb{S}^3$ obtained by gluing two Reeb foliations of the solid torus together is not analytic; i.e. the change of coordinate maps cannot be chosen to be real analytic in the transverse direction (*Hint*: Observe that it is not possible for an analytic function $f$ around $0 \in \mathbb{R}$ to have non-trivial germ exactly on one side of 0).

1.21 REMARK. Indeed, Haefliger showed that $\mathbb{S}^3$ does not admit any codimension-one analytic foliation. Hence, this exercise shows that the theory of foliations is sensitive to the regularity class.

### Exercise 4

The projection map of a fiber bundle is a submersion. We have also seen that given a submersion, there is an induced foliation on the domain whose leaves are the level sets of the submersion. In the case of a fiber bundle, this foliation is the foliation by fibers. Give an example of a submersion whose induced foliation is not a fiber bundle, and deduce that foliations from submersions are more general than fiber bundles.

## Exercise 5

a) In the proof of Theorem 1.9, show that the projection of each leaf of the constructed suspension foliation $\mathcal{F}$ to the base B is a covering map. Given a point $(x, f) \in \tilde{B} \times F$, what is the fundamental group of the leaf of $\mathcal{F}$ passing through the projection of $(x, f)$ in terms of the representation $h$?

b) Let $\pi\colon E \to B$ be the projection map of a fiber bundle with *compact* fiber F, and let $\mathcal{F}$ be a foliation such that at each point $x \in E$ the leaf of $\mathcal{F}$ passing through $x$ is transverse to the fibers and of complementary dimension. Show that for every leaf L of $\mathcal{F}$, the projection $\pi\colon L \to B$ is a covering map.

c) Give an example of a fiber bundle $\pi\colon E \to B$ with *non-compact* fiber F such that at each point $x \in E$ the leaf of $\mathcal{F}$ passing through $x$ is transverse to the fiber and of complementary dimension, but the restriction of the projection map $\pi$ to some leaf L is a not a covering map.

# 2 SOME FOUNDATIONAL THEOREMS

## 2.1 *Reeb stability theorem*

We saw in Theorem 1.3 that the holonomy of a compact leaf determines the germ of the foliation in a neighbourhood of that leaf. In particular, if the holonomy is trivial then the foliation in a neighbourhood of the compact leaf is locally the product foliation.

2.1 THEOREM (Reeb Stability). Let L be a compact leaf of a foliation $\mathcal{F}$ on a manifold M such that the image of the holonomy homomorphism

$$\mathrm{hol}\colon \pi_1(\mathrm{L}, p) \to \mathrm{G}(\mathrm{D}, p)$$

is trivial. Then L has a neighbourhood in M that is foliated as a product. In particular, this happens if L is compact and simply connected.

In codimension-one there is a global stability result.

2.2 THEOREM (Global stability). Let $\mathcal{F}$ be a transversely orientable codimension-one foliation of a compact connected manifold that has a simply connected leaf L. Then there is a submersion $f\colon \mathrm{M} \to \mathrm{S}^1$ such that the leaves of $\mathcal{F}$ are the level sets of $f$.

2.3 COROLLARY. Let $\mathcal{F}$ be a transversely orientable codimension-one foliation of a compact connected orientable 3-manifold that has a leaf homeomorphic to $\mathrm{L} = \mathrm{S}^2$ or $\mathrm{D}^2$. Then $(\mathrm{M}, \mathcal{F})$ is either $\mathrm{L} \times [0,1]$ or $\mathrm{L} \times \mathrm{S}^1$ with the product foliation.

*Proof.* If the submersion $f\colon \mathrm{M} \to \mathrm{S}^1$ is not surjective, then the level sets form the product foliation. If $f$ is surjective, then $\mathcal{F}$ is the mapping torus of $f$. Since M is orientable, $f$ is orientation-preserving. Every orientation-preserving homeomorphism (respectively diffeomorphism [Sma59]) of $\mathrm{S}^2$ or $\mathrm{D}^2$ is isotopic to the identity map. Hence $(\mathrm{M}, \mathcal{F})$ is the product foliation on $\mathrm{L} \times \mathrm{S}^1$. □

2.1 REMARK. Thurston generalised the Reeb stability theorem and showed that if the foliation is $\mathrm{C}^1$ then one can weaken the hypothesis to the following: $\mathrm{H}^1(\mathrm{L}; \mathbb{R}) = 0$ (i.e. every homomorphism $\pi_1(\mathrm{L}) \to \mathbb{R}$ is trivial) and there is no non-trivial homomorphism $\pi_1(\mathrm{L}) \to \mathrm{GL}(k, \mathbb{R})$, where $k$ is the codimension of the foliation and $\mathrm{GL}(k, \mathbb{R})$ is the general linear group. Unlike the Reeb stability, Thurston's stability is not true for $\mathrm{C}^0$ foliations.

## 2.2 *Existence of foliations*

By the Poincaré–Hopf index formula, a compact manifold M admitting a codimension-1 foliation transverse to $\partial\mathrm{M}$ satisfies $\chi(\mathrm{M}) = 0$. The converse also holds; this is a theorem of Lickorish [Lic65], Novikov [Nov65], and Zieschang

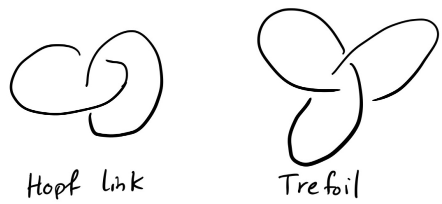


*Figure 12: The Hopf link (left) and the trefoil knot (right) are fibered.*

in dimension 3, and a deep theorem of Thurston [Thu76] in higher dimensions. Note that the condition $\chi(M) = 0$ is automatically satisfied for closed odd-dimensional manifolds. Before discussing the proof in dimension 3, we need to introduce the notions of Dehn surgery and fibered links.

A *knot* K in a 3-manifold M is a smooth embedding $K\colon S^1 \hookrightarrow M$. Similarly a *link* L is defined as a smooth embedding $L\colon \coprod S^1 \hookrightarrow M$, where $\coprod S^1$ is a disjoint union of finitely many copies of $S^1$. We often identify a knot or link with its image. Given an orientable 3-manifold M and a knot K in M, *Dehn surgery* on K is the operation of removing an open regular neighbourhood $N^\circ(K) \cong S^1 \times \mathbb{D}^2$ of K from M and attaching a solid torus to $M - N^\circ(K)$ along $\partial N(K)$ in a different way.

A link L is a compact 3-manifold M is *fibered* if the exterior $X = M - L$ of L is a 3-manifold that fibers over the circle, and moreover, the closure of the fibers are compact surfaces that intersect exactly in their common boundary L.

2.2 EXAMPLE. The unknot in the 3-sphere is a fibered knot, since its exterior fibers over $S^1$ with fiber a disc. Other examples are the Hopf link and the trefoil knot, but it is harder to visualise their fibrations over $S^1$. See Figure 12.

2.3 REMARK. Gabai [Gab86] gave a method for detecting fibered links in $\mathbb{S}^3$ using foliations.

2.4 THEOREM (Lickorish, Wallace). Every closed orientable 3-manifold can be obtained by Dehn surgery on a link in $\mathbb{S}^3$. More generally, for every compact orientable 3-manifold M there is an embedding $\Gamma \cup L \hookrightarrow \mathbb{S}^3$, where $\Gamma$ is a graph, L is a link, and M can be obtained from $\mathbb{S}^3$ by removing a tubular neighbourhood of $\Gamma$ and performing Dehn surgery on L.

2.5 THEOREM (Lickorish, Novikov, Zieschang). Any compact orientable 3-manifold with boundary a (possibly empty) union of tori admits a codimension-one transversely oriented foliation.

*Proof.* Let M a compact orientable 3-manifold with $\partial M$ a union of tori. By the theorem of Lickorish and Wallace, there is an embedding $\Gamma \cup L \hookrightarrow \mathbb{S}^3$ such that M can be obtained from $\mathbb{S}^3$ by removing a tubular neighbourhood of the graph

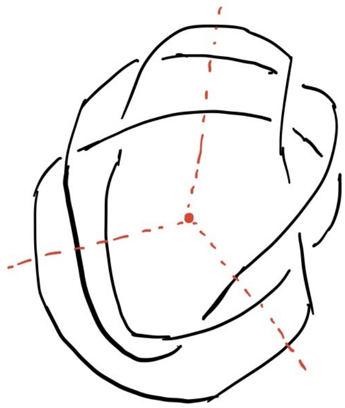

*Figure 13: The view of a braided link from a point on the z-axis with $z \gg 0$. The center point represents the z-axis and the dashed lines represent three half planes $y = rx$ transverse to the link.*

$\Gamma$ and performing Dehn surgery on the link L. Note that $\partial M$ is homeomorphic to $\partial N(\Gamma)$, and hence $\Gamma$ can be assumed to be a link. Let $\hat{L} := \Gamma \cup L$. We think of $\mathbb{S}^3$ as $\mathbb{R}^3 \cup \{\infty\}$. By Alexander's theorem, the link $\hat{L}$ can be braided, meaning that $\hat{L}$ can be isotoped such that

1. $\hat{L}$ is disjoint from the vertical line $x = y = 0$.

2. $\hat{L}$ is transverse to each half-plane $y = rx$ with $r =$ constant.

See Figure 13. Let $K \subset \mathbb{S}^3$ be the knot that is the union of the vertical line $x = y = 0$ and the point $\infty$. Then the link $R := \hat{L} \cup K$ is a fibered link, with fiber a planar surface. Hence, there is a foliation $\mathcal{G}$ on $\mathbb{S}^3 - N^\circ(R)$ whose leaves are fibers of the fibration of $\mathbb{S}^3 - N^\circ(R)$ over the circle. Use Operation 1.7 to spin the leaves of $\mathcal{G}$ around $\partial N(R)$ and obtain a foliation $\mathcal{G}_s$ of $M - N^\circ(R)$ tangential to its boundary $\partial N(R)$. Now M is obtained from $\mathbb{S}^3 - N^\circ(R)$ by attaching solid tori. Therefore, we can extend the foliation $\mathcal{G}_s$ to a foliation $\mathcal{F}$ of M by inserting a Reeb component in each of the attached solid tori.

□

## 2.3 *Exercises*

### Exercise 1

a) Let L be a compact leaf of a transversely orientable *codimension-1* foliation $\mathcal{F}$ on a manifold M such that the image of the holonomy homomorphism

$$\mathrm{hol}\colon \pi_1(\mathrm{L}, p) \to \mathrm{G}(\mathrm{D}, p)$$

is a finite group. Show that L has a neighbourhood in M that is foliated as a product (*Hint*: Show that if the germ at 0 of an orientation-preserving homeomorphism of $[0, 1)$ is periodic, then it is equal to the germ of the identity homeomorphism.).

b) Show that the conclusion does not hold in higher codimensions.

### Exercise 2

Let $\mathcal{F}$ be a *codimension-1* transversely orientable foliation of a compact connected manifold M such that all leaves of $\mathcal{F}$ are compact. Show that $(\mathrm{M}, \mathcal{F})$ is either a product or a fibration over the circle.

2.4 REMARK. A theorem of Epstein shows that if $\mathcal{F}$ is a codimension-2 foliation of a compact 3-manifold M such that all leaves of $\mathcal{F}$ are compact (i.e. circles), then $\mathcal{F}$ is a Seifert fibration. This means that each leaf L has a neighbourhood homeomorphic to a solid torus $\mathrm{S}^1 \times \mathbb{D}^2$ with $\mathrm{L} = \mathrm{S}^1 \times (0, 0)$, where the foliation on $\mathrm{S}^1 \times \mathrm{D}^2$ is induced by a periodic homeomorphism $f : \mathbb{D}^2 \to \mathbb{D}^2$ of the form

$$f(z) = \text{rotation of } z \text{ around } (0,0) \text{ by the angle } \frac{2\pi p}{q} \in 2\pi\mathbb{Q}$$

via the suspension construction. Three-dimensional manifolds admitting such foliations are called *Seifert fibered* and constitute an important class of 3-manifolds, specially in relation to the *Geometrisation Conjecture* (proved by Perelman).

### Exercise 3

Show that a compact odd-dimensional manifold M satisfies $\chi(\partial\mathrm{M}) = 2\chi(\mathrm{M})$. Deduce that a compact orientable 3-manifold M with $\chi(\mathrm{M}) = 0$ and no sphere boundary components has boundary a (possibly empty) union of tori.

### Exercise 4

Prove Alexander's theorem that every link in $\mathbb{S}^3$ (or equivalently in $\mathbb{R}^3$) can be braided.

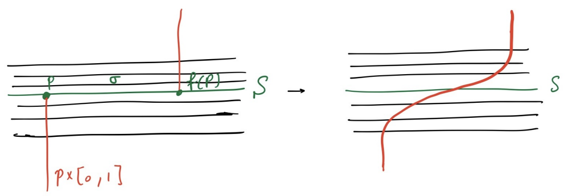


*Figure 14: Constructing a closed transversal from a transverse arc*

# 3 TAUT FOLIATIONS I

We have seen that every compact orientable 3-manifold with boundary a (possibly empty) union of tori admits a codimension-one transversely orientable foliation. Therefore, the existence of a foliation does not say much about the topology of the 3-manifold. This motivates the study of special classes of foliations that can inform us about the topology of their ambient space.

Taut foliations are a class of foliations introduced by Dennis Sullivan, which have been proved to be fruitful in the study of 3-manifolds, particularly through the work of Thurston and Gabai. We give a topological definition of taut foliations; although they have other geometric and homological characterisations as well.

3.1 DEFINITION. Let L be a leaf of a codimension-one foliation $\mathcal{F}$ on a manifold M. A *closed transversal* $\gamma$ for $\mathcal{F}$ is an immersed closed curve in M that is transverse to the foliation $\mathcal{F}$ at every point of $\gamma$. A *transverse arc* can be defined similarly.

3.2 DEFINITION (Taut foliation). A transversely orientable codimension-one foliation of a compact 3-manifold is *taut* if every leaf has a closed transversal intersecting it.

3.3 EXAMPLE. If S is a compact surface, then the product foliation on $S \times S^1$ by leaves $S \times$ point is taut, since point $\times S^1$ is a closed transversal intersecting every leaf. More generally, a foliation of a compact 3-manifold M induced by a fibration of M over $S^1$ is taut. To see this, assume that S is the fiber of the fibration $M \to S^1$. Then M is homeomorphic to the mapping torus $S \times [0,1]/(x,1) \sim (f(x),0)$ of some homeomorphism $f\colon S \to S$. Let $p \in S$ be a point, $\sigma$ be an arc in S connecting $p$ to $f(p)$, and define the closed curve $\gamma$ as the union of $p \times [0,1]$ and $\sigma$. Then one can perturb $\gamma$ to a closed transversal that intersects every leaf. See Figure 14.

A priori, the transversals for different leaves need not be the same. But as the next proposition shows, the compactness hypothesis for the ambient manifold implies that one transversal suffices for all leaves.

3.1 PROPOSITION. A transversely orientable codimension-one foliation of a connected 3-manifold is taut if and only if there is a closed transversal that intersects every leaf.

*Proof.* Denote the ambient manifold by M. For every point $p \in \mathrm{M}$, there is a closed transversal passing through $p$. To see this, let L be the leaf passing through $p$, and take a transversal intersecting L at some point $q$ and drag the intersection point $q$ to $p$ while maintaining the transversality. Since M is compact, there are finitely many such transversals $\gamma_1, \cdots, \gamma_n$ that intersect all leaves of M. Assume that $n$ is the smallest natural number with this property; we will show that $n = 1$. Define $\mathrm{M}_i$ as the union of leaves in M that intersect $\gamma_i$. Then $\mathrm{M}_i$ is an open submanifold of M and the union of $\mathrm{M}_i$ is all of $\mathrm{M}_i$. We show that any two $\mathrm{M}_i$ and $\mathrm{M}_j$ are disjoint; then the connectivity of M implies that $n = 1$. Assume to the contrary that $\mathrm{M}_i \cap \mathrm{M}_j \neq \emptyset$. We show this contradicts $n$ being minimal. Let L be a leaf intersecting both $\gamma_i$ and $\gamma_j$ at respectively $p$ and $q$. Let $\sigma$ be a path in L connecting $p$ to $q$. Then the path $\gamma_i \sigma \gamma_j \sigma^{-1}$ can be perturbed to a closed transversal $\gamma$ such that every leaf intersecting $\gamma_i$ or $\gamma_j$ intersects $\gamma$. This shows that $n$ is not minimal. The contradiction completes the proof. □

The next proposition shows that non-compact leaves of codimension-one foliations of compact manifolds always have closed transversals.

3.2 Proposition. Let L be a non-compact leaf of a transversely orientable codimension-one foliation of a compact manifold M. Then L has a closed transversal.

*Proof.* The ambient manifold can be covered by foliation charts, where the the restriction of the foliation to each chart is a product. Since M is compact, finitely many foliation charts suffices to cover M. Since L is non-compact, it should intersect at least one of the charts infinitely many times. In particular, there is a transverse arc $\gamma$ in this chart with both endpoints on L. Let $\alpha$ be an arc in L connecting the endpoints of $\gamma$. Then $\alpha \cup \gamma$ can be slightly perturbed to form a closed transversal for L; here we use the hypothesis that the foliation is transversely orientable. □

3.4 exercise. Show that Proposition 3.2 holds without the transverse orientability hypothesis as well.

Next we talk about some non-examples.

3.5 example. Let $\mathcal{F}$ be a foliation of a closed 3-manifold M such that $\mathcal{F}$ contains an embedded copy of a Reeb foliation of the solid torus $\mathrm{S}^1 \times \mathrm{D}^2$. We allow the core of the solid torus to be knotted in M. Then $\mathcal{F}$ is not taut. To see this, without loss of generality assume that the transverse orientation of the boundary torus leaf of $\mathrm{S}^1 \times \mathrm{D}^2$ points into the 3-dimensional submanifold $\mathrm{S}^1 \times \mathrm{D}^2$. Hence there could be no closed transversal intersecting this torus leaf.

3.6 EXAMPLE (Generalised Reeb component). Let $S$ be a compact orientable surface with $\partial S \neq \emptyset$. Let $M = S \times S^1$, and denote the product foliation on $M$ by $\mathcal{G}$. The restriction of the foliation $\mathcal{G}$ to each component of $\partial M$ is the product foliation of a torus by circles. Choose an orientation on the $S^1$ factor of $M = S \times S^1$; this induces a transverse orientation on the leaves of $\mathcal{G}$. Spin the leaves of $\mathcal{G}_{|\partial M}$ according to this orientation of $S^1$ to obtain a foliation $\mathcal{F}$ of $M$ tangent to $\partial M$. Note that the transverse orientation on $\partial M$ either entirely points into $M$ or entirely points out of $M$. Moreover when $S$ is a disc, the foliation $\mathcal{F}$ is the Reeb foliation on $M = D^2 \times S^1$. This is called the generalised Reeb component. Taut foliations cannot contain generalised Reeb components, since all boundary components point to the inside or all point to the outside.

The above examples motivate the following definition.

3.7 DEFINITION (Dead end component). Let $\mathcal{F}$ be a transversely oriented codimension-one foliation of a closed 3-manifold $M$. An embedded submanifold $N$ of $M$ is a *dead end component* if $\partial N$ is a non-empty union of leaves and the transverse orientation of $\mathcal{F}$ restricted to $\partial N$ entirely points into $N$ or entirely points out of $N$.

Note that a foliation having a dead end component cannot be taut. Later we will see that the converse holds as well: a foliation of a 3-manifold is taut if and only if it contains no dead end component.

3.8 REMARK. It follows from the Poincaré–Hopf formula that $\chi(\partial N) = 0$. Using the global Reeb stability theorem, it follows that $\partial N$ is a union of tori and Klein bottles. If $M$ is orientable as well, the leaves are orientable and hence $\partial N$ is a union of tori.

3.3 PROPOSITION. Let $L$ be a compact leaf of a taut foliation of a closed 3-manifold $M$. Then $L$ is homologically non-trivial in $H_2(M; \mathbb{Z})$.

*Proof.* Otherwise, a homologically trivial compact leaf bounds a dead end component. □

3.9 EXAMPLE. Let $M$ be a taut foliation of an integer homology 3-sphere, meaning that the homology groups of $M$ with integer coefficients are isomorphic to those of the 3-sphere. In particular $H_2(M; \mathbb{Z}) = 0$. Then all leaves of $\mathcal{F}$ are non-compact.

3.10 REMARK. Some but not all homology 3-spheres admit taut foliations. There is a conjectural (at the time of this writing) characterisation of rational homology 3-spheres that admit taut foliations, known as the *L-space conjecture*. On the other hand, Gabai proved that every compact orientable irreducible 3-manifold with boundary a (possibly empty) union of tori and satisfying $H_2(M, \partial M; \mathbb{R}) \neq 0$ admits a taut foliation. The foliations constructed by Gabai have a compact leaf, which is consistent with the condition $H_2(M, \partial M; \mathbb{R}) \neq 0$.

## 3.1 *Prime decomposition for 3-manifolds*

Let $M_1$ and $M_2$ be connected oriented 3-manifolds. Then we can form their *connected sum*, denoted by $M_1 \sharp M_2$ as follows. Choose smoothly embedded closed 3-balls $i_1 : \mathbb{B} \hookrightarrow M_1$ and $i_2 : \mathbb{B} \hookrightarrow M_2$, and denote the image of $i_1$ and $i_2$ by respectively $B_1$ and $B_2$. Glue $M_1 - \mathrm{int}(B_1)$ to $M_2 - \mathrm{int} B_2$ by an orientation-reversing diffeomorphism $i \colon \partial B_1 \to \partial B_2$ to obtain $M_1 \sharp M_2$. The connected sum operation does not depend on the choices involved; see Exercise 6. Note that for any 3-manifold M the connected sum $M \sharp \mathbb{S}^3$ is diffeomorphic to M.

3.11 DEFINITION (Prime 3-manifold). A connected oriented 3-manifold M is *prime* if it is not diffeomorphic to any connected sum $M_1 \sharp M_2$ with $M_i \not\cong \mathbb{S}^3$.

In 1928, Kneser showed that each compact connected oriented 3-manifold can be written as a connected sum

$$M = M_1 \sharp M_2 \sharp \cdots \sharp M_k,$$

where each $M_i$ is prime; the point being that one cannot carry on writing each summand as a sum of non-trivial factors. In 1962, Milnor showed that the above decomposition of M is unique up to reordering of factors [Mil62]. This is called the *prime decomposition* of M.

3.12 DEFINITION (Irreducible 3-manifold). A 3-manifold M is *irreducible* if every smoothly embedded two-dimensional sphere $S \hookrightarrow M$ bounds an embedded 3-dimensional ball on at least one side.

The notions of prime and irreducible 3-manifold almost coincide with one exception.

3.4 PROPOSITION. A compact connected oriented 3-manifold $M \neq S^2 \times S^1$ is prime if and only if it is irreducible.

*Proof.* Note that a 3-manifold M is prime if and only if every separating sphere in M bounds a 3-ball, where we think of the separating 2-sphere as the sphere in the connected sum operation. Therefore every irreducible 3-manifold is prime.

Now let M be a prime manifold that is not irreducible. We show that $M = S^2 \times S^1$. Since M is not irreducible, there is an embedded smooth sphere S in M that does not bound a 3-ball. Since M is prime, S is non-separating. Let $\alpha \subset M$ be a simple closed curve intersecting S once. Let N be a tubular neighbourhood of $S \cup \alpha$. Then $\partial N$ is a separating 2-sphere, and so $\partial N$ bounds a 3-ball B, necessarily to the outside of N. Note that we have $M = N \cup_{\partial N} B$. By Smale's theorem, every orientation-reversing diffeomorphism of a 2-sphere is isotopic to the identity. Hence the diffeomorphism type of $N \cup_{\partial N} B$ is uniquely determined. Therefore, it is enough to exhibit one gluing of N to B along their boundaries resulting in $S^2 \times S^1$. Write S as the union of two discs $D \cup D'$ and B as $D' \times I$. Then by gluing $B = D' \times I$ to $N = N(\alpha \cup S)$, we see that $M = N \cup_{\partial N} B$

is diffeomorphic to $S^2 \times S^1$. □

3.5 Proposition. The manifold $S^2 \times S^1$ is prime but not irreducible.

*Proof.* Let $S^2 \times S^1 = M_1 \sharp M_2$. We show that one of $M_1$ and $M_2$ is $\mathbb{S}^3$. In other words, if $S \subset S^2 \times S^1$ is the attaching 2-dimensional sphere between ($M_1$ − 3-ball) and ($M_2$ − 3-ball), then S bounds a 3-ball in $S^2 \times S^1$. Note that S is *separating* in $S^2 \times S^1$ by the definition of the connected sum. By Seifert–Van Kampen theorem, we have $\mathbb{Z} \cong \pi_1(S^2 \times S^1) = \pi_1(M_1) * \pi_1(M_2)$. Therefore, one of $M_1$ and $M_2$ have trivial fundamental group, otherwise $\mathbb{Z} = \pi_1(M_1) * \pi_1(M_2)$ would be non-abelian. Assume that $\pi_1(M_1)$ is trivial and so $M_1$ is simply connected. Therefore $M_1$ lifts to the universal cover $S^2 \times \mathbb{R}$ of $S^2 \times S^1$. Note that $S^2 \times \mathbb{R}$ can be identified with $\mathbb{R}^3 - \{0\}$ by looking at concentric spheres around the origin in $\mathbb{R}^3$. Hence we can think of $M_1$ as embedded in $\mathbb{R}^3 - \{0\} \subset \mathbb{R}^3$. By Alexander's theorem, the sphere $S = \partial M_1$ bounds an embedded 3-ball on at least one side. This has to be the side not containing the origin and so ($M_1$ − 3-ball) is diffeomorphic to a ball, implying that $M_1 = \mathbb{S}^3$. This shows that $S^2 \times S^1$ is prime. Clearly $S^2 \times S^1$ is not irreducible since $S^2 \times$ point is a non-separating sphere. □

## 3.2 *Taut foliations and irreducibility*

The following theorem of Alexander is one of the first applications of foliations in studying 3-manifolds.

We need the notion of a Morse function for the proof. Let M be a smooth manifold of dimension $n$, and $f\colon M \to \mathbb{R}$ be a real valued smooth function. A point $p$ of M is a critical point, if the first derivative of $f$ at $p$ vanishes. A critical point $p$ of $f$ is *non-degenerate* if the matrix of the second derivatives of $f$ at $p$ (i.e. the Hessian matrix) is non-singular. The *index* of a non-degenerate critical point $p$ is the number of negative eigenvalues of the Hessian matrix at the point $p$; this is intuitively the number of directions around $p$ at which $f$ decreases. A smooth function $f\colon M \to \mathbb{R}$ is a *Morse function* if all critical points of $f$ are non-degenerate.

The *Morse Lemma* states that if $p$ is a non-degenerate critical point of $f\colon M \to \mathbb{R}$ then there is a chart $(x_1, \cdots, x_n)$ in a neighbourhood of $p$ such that $p$ corresponds to the point $x_i = 0$ and the function $f$ has the form

$$f(x) = f(p) - x_1^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_n^2,$$

where $r$ is the index of $p$. Note that the Morse Lemma in particular implies that non-degenerate critical points are isolated.

Given a smooth manifold M, the set of Morse functions $f\colon M \to \mathbb{R}$ forms an open and dense subset of the set of all smooth functions $f\colon M \to \mathbb{R}$ with the $C^2$ topology; hence Morse functions are generic and every smooth function can be approximated by Morse functions.

3.6 Theorem (Alexander). The manifold $\mathbb{R}^3$ is irreducible.

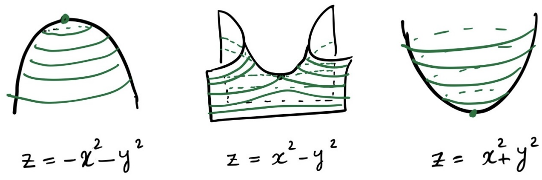


*Figure 15: Three types of singularities for the induced foliation on* S. *Here the leaves of the singular foliation are the level sets of the height function.*

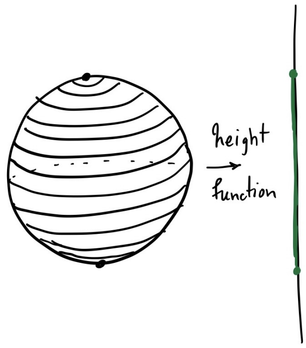


*Figure 16: A sphere in standard position in* $\mathbb{R}^3$

*Proof.* Consider the foliation $\mathcal{F}$ of $\mathbb{R}^3$ by horizontal planes $z$ = constant. Let S be a sphere in $\mathbb{R}^3$, and isotope S slightly such that the height function restricted to S is a Morse function such that the critical points of $f$ have distinct heights. This can be done, for example, by applying a small rotation to S. See Figure 15 for the three possible types of critical points. Let $\mathcal{F}|\mathrm{S}$ be the induced singular foliation on S. Hence $\mathcal{F}|\mathrm{S}$ has only finitely many singularities of center or saddle type. If the number of centers and saddles are respectively $\mathrm{I}_+$ and $\mathrm{I}_-$, then by Poincaré–Hopf formula we have

$$2 = \chi(\mathrm{S}) = \mathrm{I}_+ - \mathrm{I}_-.$$

In particular, $\mathrm{I}_+ > 0$, and so there is at least one center. Alternatively, this could be seen by looking at a point $p \in \mathrm{S}$ with maximal (or minimal) height.

The idea is to use the induced foliation $\mathcal{F}|\mathrm{S}$ as a road map to simplify S via isotopy until it is in the *standard position*, i.e. with only two center singularities and with no saddle singularities. See Figure 16. We argue by induction on the total number of singularities of $\mathcal{F}|\mathrm{S}$. An important property of the foliation $\mathcal{F}|\mathrm{S}$ is that it has trivial holonomy; i.e. every circle leaf of $\mathcal{F}|\mathrm{S}$ has a neighbourhood consisting of parallel circles.

Let $p$ be a center tangency. The induced foliation on S in a small neighbourhood of $p$ consists of concentric circles. Take a maximal neighbourhood U of $p$ in S that consists of concentric circles with no singularities on them. Then U is an open disc, otherwise it could be extended to a larger such neighbourhood. If $\overline{\mathrm{U}} = \mathrm{S}$, then S is in the standard position. Otherwise, there should be at

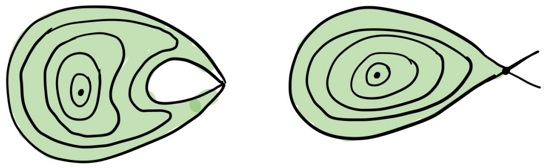

*Figure 17: The two possibilities for the closure of a 'maximal' neighbourhood of a center singularity.*

least one saddle singularity, say $q$, on $\overline{U} - U$. In fact $q$ is the only singularity on $\overline{U} - U$ since we assumed that distinct critical points have distinct heights. Locally there are 4 rays of the singular foliation $\mathcal{F}_{|S}$ coming out of $q$. Note that if $h$ is the height function, then all these rays lie inside the compact level set $h^{-1}(h(q))$ which is a graph embedded in S; in particular the 4 rays coming out of $q$ have to close up globally.

There are two possibilities for the shape of $\overline{U} - U$ as in Figure 17. The relative position of S with respect to the foliation $\mathcal{F}$ is shown in a neighbourhood of $\overline{U}$ in each case. In either case, there is an isotopy of S that simplifies the induced foliation on S, and reduces the number of centers and saddles each by one. Repeating this procedure, S can be isotoped such that no saddle singularities are left, and the induced foliation on S has exactly two center singularities. Then, each circle of $\mathcal{F}|S$ bounds a disc in $\mathbb{R}^3$, and it is not hard to see that these discs patch together to form an embedded ball bounded by S on one side of it.

□

3.13 REMARK. Alexander [Ale24] proved the above theorem for PL embedded spheres in $\mathbb{R}^3$.

Rosenberg [Ros68] generalised Alexander's argument to 3-manifolds admitting taut foliations.

3.7 THEOREM (Rosenberg). Let $M \neq S^2 \times S^1$ be a compact orientable 3-manifold that admits a taut foliation. Then M is irreducible.

3.14 EXAMPLE. Rosenberg's theorem implies that any compact orientable 3-manifold M fibering over $S^1$, with fiber not the sphere, is irreducible.

3.15 REMARK. Prior to Rosenberg, Novikov [Nov65] had shown that for manifolds as in Theorem 3.7, the second homotopy group $\pi_2(M)$ is trivial; i.e. every immersed sphere bounds a (not necessarily embedded) 3-ball. Rosenberg's proof relies on the work of Novikov.

## 3.3 *Exercises*

### Exercise 1

Let M be the connected sum of 3-manifolds $M_1$ and $M_2$. Show that the fundamental group $\pi_1(M)$ is isomorphic to $\pi_1(M_1) * \pi_1(M_2)$. Deduce that if $M_1$ is not simply connected, then it has no inverse; i.e. there is no $M_2$ such that $M_1 \sharp M_2$ is diffeomorphic to $\mathbb{S}^3$.

### Exercise 2

Let S be a non-separating sphere in a connected oriented 3-manifold M. Show that M can be written as $(S^2 \times S^1)\sharp N$ for some oriented 3-manifold N.

### Exercise 3

Prove the last statement in the proof of Alexander's theorem that if $\mathcal{F}|S$ has exactly two center singularities and no saddle singularities, then S bounds an embedded ball on one side of it.

### Exercise 4

Use Alexander's theorem to show that $\mathbb{S}^3$ is irreducible.

### Exercise 5

Use Alexander's theorem to deduce that if $K \subset \mathbb{S}^3$ is a knot with tubular neighbourhood N(K), then its complement $X := \mathbb{S}^3 - N^\circ(K)$ is irreducible.

For the next exercise, you need the notion of isotopy defined below.

3.16 DEFINITION (Isotopy). Given manifolds N and M and two embeddings $f_0, f_1 \colon N \to M$, we say that $f_1$ and $f_2$ are *isotopic* if there is a continuous map $F \colon N \times [0,1] \to M$ such that for each $t \in [0,1]$ the restriction $F_{|N\times\{t\}}$ is an embedding and that $F_{|N\times\{0\}} = f_0$ and $F_{|N\times\{1\}} = f_1$.

### Exercise 6

Let $M_1$ and $M_2$ be compact orientable 3-manifolds. Show that the homeomorphism type of the connected sum $M_1 \sharp M_2$ does not depend on the choice of the embeddings $i_1 \colon \mathbb{B} \hookrightarrow M_1$ and $i_2 \colon \mathbb{B} \hookrightarrow M_2$ and the gluing homeomorphism $i \colon \partial B_1 \to \partial B_2$. You can use the following two facts without proof:

1. Smale's theorem [Sma59]: every orientation-preserving diffeomorphism of a two-dimensional sphere is isotopic to the identity map.

   This implies that the isotopy class of the map $i$ is unique.

2. Palais' theorem [Pal60]: if $j_1 \colon \mathbb{B}^n \hookrightarrow M^n$ and $j_2 \colon \mathbb{B}^n \hookrightarrow M^n$ are two embeddings of the closed $n$-ball into a connected $n$-dimensional manifold M, then $i_1$ and $i_2$ are isotopic.

   This implies that the isotopy classes of $i_1$ and $i_2$ are unique.

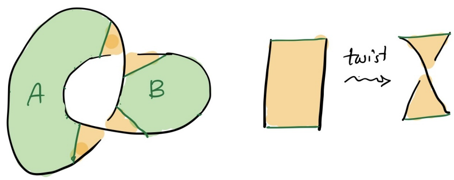


*Figure 18: A Seifert surface for the Hopf link (left) obtained by two disks* A *and* B *that are attached by two twisted bands (right).*

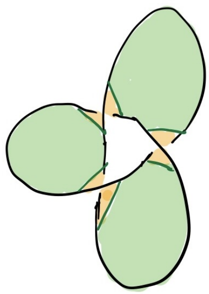

*Figure 19: A non-orientable spanning surface for the trefoil.*

# 4 TAUT FOLIATIONS II

In this section, we discuss some of the applications of taut foliations, mostly without proof.

## 4.1 *Knot genus*

Given a link $L$ in $\mathbb{R}^3$, there is an embedded compact orientable surface $S$ in $\mathbb{R}^3$ with $S \cap L = \partial S = L$. See Figure 18 for the Hopf link.

Note that a non-orientable spanning surface is not a Seifert surface. See Figure 19 for an example of a Möbius band spanning the trefoil. However a Seifert surface can always be constructed from a projection of the knot via Seifert's algorithm. The algorithm is as follows:

1. Orient the knot. Resolve each crossing according to orientation as in Figure 20 to obtain a union of disjoint circles. Each such circle bounds a disc in the plane of the knot projection.

2. Locate the discs in different heights and attach them to each other by twisted bands. See Figure 21.

4.1 Activity. a) Using paper, scissor, and tape, build a Seifert surface for the trefoil knot via Seifert's algorithm (*Tip*: long thin bands are easier to twist in practice).

*Figure 20: Seifert algorithm I: resolving the crossings as in the right hand side produces a union of disjoint circles.*

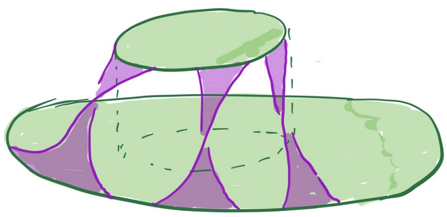

*Figure 21: A Seifert surface for the trefoil produced by Seifert's algorithm.*

b) A compact orientable surface S with one boundary component is homeomorphic to a torus with a disc removed if and only if there are disjoint properly embedded arcs $\alpha$ and $\beta$ in S such that cutting S along $\alpha \cup \beta$ produces a disc. Use this fact to show that the surface obtained by Seifert's algorithm applied to the trefoil knot is homeomorphic to a torus with a disc removed.

4.1 REMARK. A knot K in a 3-manifold M has a Seifert surface if and only if $[K] = 0 \in H_1(M;\mathbb{Z})$. Note that this condition is automatically satisfied if M is an integer homology 3-sphere.

4.2 DEFINITION (Seifert surface). Given a knot K in a 3-manifold M, a *Seifert surface* for K is any embedded compact orientable surface S in M such that $S \cap K = \partial S = K$.

4.3 DEFINITION (Knot genus). Given a knot K in $\mathbb{R}^3$, the *genus* of K, denoted by $g(K)$, is defined as the minimum genus of a Seifert surface for K.

4.4 EXAMPLE. A knot K in a 3-manifold M is called the *unknot* if K can be isotoped to a round circle standardly embedded in a small 3-ball in M. Then a knot K has genus 0 if and only if it is the unknot.

4.5 EXAMPLE. The Hopf link has genus 0 since it bounds an annulus. The trefoil knot has genus one since it bounds a genus one surface and it is not isotopic to the unknot (we do not give a proof of this fact). Note that the genus of the Seifert surface in Figure 21 can be read by calculating its Euler characteristic and using the following formula

$$\chi(S) = 2 - 2g - b,$$

where $g$ is the genus and $b = 1$ is the number of boundary components.

Note that in order to prove an upper bound $g(K) \leq C$ it is enough to exhibit one Seifert surface with genus at most C. However, proving a lower bound $c \leq g(K)$ requires ruling out the existence of any Seifert surface with genus less than $c$. In general there are infinitely Seifert surfaces, up to isotopy, for a knot K (for example one can locally connect sum a Seifert surface with a torus), and so proving a lower bound is the challenging part. We will see that taut foliations can be used to give such lower bounds.

4.6 REMARK. Agol, Hass, and Thurston [AHT06] proved that the problem of upper bound for knot genus in closed orientable 3-manifolds is **NP**-complete: Here the input is a triangulated closed orientable 3-manifold M, a knot K which is a union of edges of the triangulation, and an integer $n$, and the problem asks whether $g(K) \leq n$?

## 4.2 *Thurston norm*

Let K be a knot in the 3-sphere. Let N(K) be a tubular neighbourhood of K and $X := \mathbb{S}^3 - N^\circ(K)$ be the exterior of an open tubular neighbourhood of K. Then X is a compact orientable 3-manifold with boundary the torus $T = \partial N(K)$. Define the *meridian* $\mu$ for K as the boundary of a disc $\mathrm{pt} \times D^2$ in $S^1 \times D^2 \cong N(K)$. Given a Seifert surface S for K, define the surface $S'$ by restricting S to X; i.e. $S' = S \cap X$. If the tubular neighbourhood N(K) is small, then $\partial S \subset T$ is a curve that intersects a meridian $\mu$ exactly once and transversely. Fix an orientation on K and orient S such that the boundary orientation on $\partial S$ agrees with that of K. We claim that the homology class of $S'$ in $H_2(X, \partial X)$ does not depend on the choice of the Seifert surface S. To see this note that

$$H^1(X) = H^1(\mathbb{S}^3 - N^\circ(K)) \cong H^1(\mathbb{S}^3 - K) \cong H_1(K) \cong \mathbb{Z},$$

where we used Alexander duality in $H^1(\mathbb{S}^3 - K) \cong H_1(K)$. On the other hand by Poincaré duality we have

$$H_2(X, \partial X) \cong H^1(X).$$

Combining them we see that $H_2(X, \partial X) \cong \mathbb{Z}$ generated by a surface Poincaré dual to the meridian of K. On the other hand any surface $S'$ constructed as above intersects the meridian exactly once and positively, and so it is Poincaré dual to $\mu$. Therefore, all such $S'$ are homologous in $H_2(X, \partial X)$. Denote this common homology class by $h \in H_2(X, \partial X)$.

It follows that the genus of K can be defined as the minimum genus of compact connected orientable surfaces in the homology class of $h \in H_2(X, \partial X)$. This perspective was used by Thurston to define a norm on the second homology group of compact orientable 3-manifolds, 'generalising' the notion of knot genus. Thurston used with a variant of the Euler characteristic instead of genus.

4.7 DEFINITION. Given a compact orientable surface S with components $S_1, \cdots, S_n$, define the *negative part of the Euler characteristic*, $\chi_-(S)$, as sum of $|\chi(S_i)|$ over those $S_i$ that have negative Euler characteristic. Equivalently $\chi_-(S)$ is the absolute value of the Euler characteristic after discarding any sphere and disc components.

4.8 DEFINITION (Thurston norm). Let M be a compact orientable 3-manifold. Given an integral homology class $a \in H_2(M, \partial M; \mathbb{R})$, define the *Thurston norm* of $a$, denoted by $x(a)$, as

$$x(a) = \min\{\chi_-(S) | S \subset M \text{ properly embedded compact orientable surface}, [S] = a\}.$$

4.9 REMARK. We will see later that this definition can be extended to a continuos function $x\colon H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}^{\geq 0}$, which is a semi-norm; i.e.

1. (non-negativity) $x(a) \geq 0$ for every $a$; and
2. (triangle inequality) $x(a+b) \leq x(a) + x(b)$ for every $a, b$.

Note that a semi-norm is a norm if it further satisfies $x(a) = 0 \iff a = 0$.

Thurston showed that compact leaves of taut foliations are norm-minimizing. This was used by Gabai to compute the genus of all knots with at most 10 crossings [Gab84]; this is done by identifying a minimal genus candidate S and then constructing a taut foliation having S as a compact leaf.

4.2 THEOREM (Thurston). Let M be a compact orientable 3-manifold with boundary a (possibly empty) union of tori. Let $\mathcal{F}$ be a taut foliation of M such that the restriction of $\mathcal{F}$ to each torus boundary component is a suspension of a circle homeomorphism. Then every compact leaf S of $\mathcal{F}$ is norm-minimizing; i.e. $x([S]) = \chi_-(S)$.

4.10 EXAMPLE. Let K be a fibered knot in a closed orientable 3-manifold M, with fiber F. Recall that this means the exterior of a tubular neighbourhood of K is the total space of a fibration over $S^1$ such that each fiber intersects the meridian of K algebraically non-zero number of times. The fibration is a taut foliation, and the fiber F is a compact leaf of a taut foliation. Therefore the fiber F is norm-minimizing. Since $\chi_-(F) = 2g(F) - 1$ if $g(F) \geq 1$, and $\chi_-(F) = 0$ otherwise, it follows that F is a minimum genus Seifert surface for K.

As an example, the trefoil is a fibered knot with fiber a surface of genus one with one boundary component. Therefore the genus of trefoil is equal to one.

We recall the proof of Alexander theorem stating that every smoothly embedded sphere in $\mathbb{R}^3$ bounds an embedded 3-ball. We considered the foliation $\mathcal{F}$ of $\mathbb{R}^3$ by parallel planes and looked at the induced singular foliation on S. We then used the induced foliation $\mathcal{F}|S$ as a road map to inductively simplify $\mathcal{F}|S$ via isotopies of S until $\mathcal{F}|S$ has exactly two centres and no saddles; at which point the surface S is in standard position in $\mathbb{R}^3$ and so bounds a 3-ball. In the process of simplifying $\mathcal{F}|S$, a centre was cancelled together with a saddle; however sometimes a new circle tangency was introduced. Roussarie [Rou74] and Thurston generalised this argument to show the following, which is the main ingredient in the proof of Theorem 4.2. An immersed connected surface S in a compact 3-manifold is $\pi_1$*-injective* if the map $i_*\colon \pi_1(S) \to \pi_1(M)$, induced by inclusion, is injective. A proof of Roussarie–Thurston theorem can be found in [CC00b] and [Gab00].

4.3 Theorem (Roussarie, Thurston). Let M be a compact orientable 3-manifold with boundary a (possibly empty) union of tori. Let $\mathcal{F}$ be a taut foliation on M such that the induced foliation on each boundary component of M is a suspension of a circle homeomorphism. Let S be a connected embedded $\pi_1$-injective surface in M with $\chi(S) \leq 0$. Then S can be isotoped such that the induced foliation on $\mathcal{F}$ has only finitely many saddle and circle singularities and no centre singularity.

4.11 remark. a) A foliation of a 2-dimensional torus is a suspension of a circle homeomorphism if and only if it contains no 2-dimensional Reeb component.

b) Unlike the foliation of $\mathbb{R}^3$ by horizontal planes, $\mathcal{F}$ can have holonomy and one needs to take more care in the proof.

c) The hypothesis $\chi(S) \leq 0$ is necessary: if S is a sphere or disc, then the induced foliation on S has at least one centre singularity by the Poincaré–Hopf formula.

d) The hypothesis of S being $\pi_1$-injective is necessary. For example let $\mathcal{F}$ be a product foliation (or any taut foliation) and let S be a closed surface of genus $\geq 1$ such that S is standardly embedded in a small ball in M. Then one can not remove the centre singularities by any isotopy of S (we do not prove this, but the statement should be intuitively plausible to the reader).

Gabai proved a converse to Theorem 4.2, namely a norm-minimising surface is a compact leaf of a taut foliation. This shows that taut foliation and the minimum genus problem are intimately related to each other.

4.4 Theorem (Gabai). Let M be a compact orientable irreducible 3-manifold with boundary a (possibly empty) union of tori such that $H_2(M, \partial M; \mathbb{R})$ has rank at least one. Let S be a properly embedded compact orientable surface in

M that is norm-minimizing. There is a taut foliation $\mathcal{F}$ on M such that

i) S is a union of compact leaves of $\mathcal{F}$; and

ii) the induced foliation on each boundary component of M is a suspension of a circle homeomorphism.

4.12 REMARK. The irreducibility hypothesis is necessary: by Rosenberg a compact orientable 3-manifold $M \neq S^2 \times S^1$ admitting a taut foliation is irreducible.

## 4.3 *Exercises*

### Exercise 1

Prove that Seifert's algorithm produces an *orientable* surface spanned by the knot in $\mathbb{S}^3$.

### Exercise 2

Show that a compact orientable surface S with one boundary component is homeomorphic to a torus with a disc removed if and only if there are disjoint properly embedded arcs $\alpha$ and $\beta$ in S such that cutting S along $\alpha \cup \beta$ produces a disc.

### Exercise 3

Let M and $\mathcal{F}$ be as in the hypothesis of Roussarie–Thurston theorem. Let T be a properly embedded $\pi_1$-injective torus or annulus in M. Use the statement of Roussarie–Thurston theorem to deduce that T can be isotoped such that the induced foliation on T has no saddle and centre singularities.

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