LTCC Advanced Course

Title: Gowers norms and the Möbius function

Basic Details:

- Core Audience: 2nd-3rd year, pure, but 1st years very strongly encouraged to attend!
- Course Format: 5 x 2 hours lectures.

Course Description:

- Keywords: analytic number theory, circle method, nilmanifolds, bilinear forms, multiplicative functions, harmonic analysis.

- Syllabus:
  - Lecture 1: The classical picture, circle method, Type I/II sums, three-term arithmetic progressions weighted by Möbius.
  - Lecture 2: Higher-order obstructions, Gowers norms and their basic properties. Statement of the inverse theorem for Gowers norms (without definition of nilsequence), statement of strong orthogonality, a generalized von Neumann theorem and deduction of k-term arithmetic progression cancellation.
  - Lecture 3: Nilsequences, definitions and examples.
  - Lecture 4: Statement of the factorization theorem for nilsequences. Proof of the abelian case, sketch proof of quadratic case and general case.
  - Lecture 5: Proof of strong orthogonality. Survey of current state of the field.

- Abstract:
  Many arguments in analytic number theory use Fourier analysis to study the distribution of certain sets of integers. For example, a classical argument from the 1930s uses analysis of characters on the integers to show that there are infinitely many three-term arithmetic progressions of primes. In the last 25 years, a new theory (loosely termed 'higher order Fourier analysis') has been created, which extends and generalises some of the classical techniques. The purpose of this course is to introduce students to some of the main ideas in both the classical and extended theories, in particular proving an application involving k-term arithmetic progressions that lies beyond the classical picture. Although at heart this is a course in analysis and number theory, some nilpotent algebra and combinatorics will also take a central role.

- Recommended reading:
  - Chapters 11.1 and 11.2 of *T. Tao and V. Vu, Additive Combinatorics, CUP 2006.*
  - Chapters 24, 25, 26 of *H. Davenport, Multiplicative Number Theory (Third Edition), Springer 2000.*
  - Section 8 of *Green, B. and Tao, T., 2010. Linear equations in primes. Annals of mathematics, pp.1753-1850.*

- Additional Optional reading:
  - Green, B. and Tao, T., 2010. Linear equations in primes. *Annals of mathematics*, pp.1753-1850.
  - Green, B. and Tao, T., 2012. The quantitative behaviour of polynomial orbits on nilmanifolds. *Annals of Mathematics*, pp.465-540.
  - Green, B. and Tao, T., 2012. The Möbius function is strongly orthogonal to nilsequences. *Annals of Mathematics*, pp.541-566.

– Prerequisites: Nothing beyond a general undergraduate mathematics education, though the material will be quite varied. There will be some group theory, some Fourier analysis, some linear algebra, and a large quantity of elementary arguments.

Format:

- No. of discussion sheets: 4
- Electronic lecture notes: Yes, available at the start of the course.

Lecturer Details:

- Lecturer: Dr. Aled Walker
- Lecturer home institution: King's College London
- Lecturer e-mail: aled.walker@kcl.ac.uk
- Lecturer telephone number: +44 7799745845