London Taught Course Centre

2023 Examination

Graph Theory

Instructions to candidates

This open-book exam has 3 questions. Some parts are harder than others. Substantial credit will be given for partial answers and ideas which you cannot justify, provided that you clearly distinguish between statements which you believe but do not see how to prove, statements which you believe you have proved, and statements you think are obvious enough not to need a proof.

You may wish to use Internet searches in addition to the lecture notes. This is allowed and you may use results you find there (remember though that you should show own work for each question), but you should reference your sources.

Question 1

(a) Show that the set of all trees, with subgraph ordering $\leq_S$ as the ordering relation, is not a well-quasi-ordering.

Let G, H be graphs so that H is obtained from G by a sequence of suppressions of vertices of degree two.

(b) Show that for all $k \geq 3$, if H is k-choosable, then G is k-choosable.

(c) Give an example showing that we cannot take $k = 2$ in part (b).

Question 2

Theorem 7 from Lecture 4 reads as follows: "$1/n$ *is a threshold for* $G(n,p)$ *to contain a triangle.*" Generalise this theorem to give a threshold for $G(n,p)$ to contain a clique K_r, and give a proof.

Question 3

Use the regularity lemma to prove that for any $\gamma, \nu > 0$ there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that if a graph G on $n \geq n_0$ vertices has minimum degree $(1/2 + \gamma)n$ and every set of vertices of size ηn contains an edge, then G contains a collection of vertex-disjoint triangles covering at least $(1-\nu)n$ vertices of G.

To do this, execute the following steps in order.

1. Apply the regularity lemma with parameters $\varepsilon < \gamma^{1000} \times \nu^{1000}$ (to be read: ridiculously small compared to any reasonable function of both γ and ν) and $k_0 = 1/\varepsilon$. Consider the cluster graph $R(G)$ as defined in the lectures, except only take edges of weight at least $\gamma/4$.

   Show that this cluster graph has minimum degree at least $|R(G)|/2$.

2. Is there a result that guarantees a perfect matching in $R(G)$? Can we reduce the problem of finding vertex-disjoint triangles to a single regular pair (V_i, V_j)?

3. Show that if (U, V) is an ε-regular pair from the partition of density at least $\gamma/4$, and every set of ηn vertices contains an edge, then if η is sufficiently small compared to ε, we can carry out the following procedure iteratively, covering all but $2\varepsilon|U|$ and $2\varepsilon|V|$ vertices respectively on each side: Pick $v \in U$ of maximum degree to V; find an edge in its neighbourhood in V; remove the resulting triangle; switch sides.

END OF PAPER

