London Taught Course Centre: Graph Theory
Exam Solutions
2023

Question 1

(a) Show that the set of all trees, with subgraph ordering $\leq_S$ as the ordering relation, is not a well-quasi-ordering.

Solution: For $k \geq 1$, let T_k be the tree formed by taking a path with $k+1$ vertices and then adding four new vertices, two of whom are adjacent to each of the end vertices of the path. We claim that for $k \neq \ell$, T_k is not a subgraph of T_ℓ. The easiest way to see this is by observing that every proper connected subgraph of T_ℓ has at most one vertex of degree 3, while T_k has two vertices of degree 3.

Let G, H be graphs so that H is obtained from G by a sequence of suppressions of vertices of degree two.

(b) Show that for all $k \geq 3$, if H is k-choosable, then G is k-choosable.

Solution: Suppose $k \geq 3$ and H is k-choosable. Let L be a list assignment of k colours to each vertex of G. Each vertex of H corresponds to a vertex of G. Let L_H be list assignment to the vertices of H corresponding to the lists given to the vertices of G. Then H can be properly coloured using colours from each of the vertices' lists. This corresponds to a partial colouring of G (proper, since if two vertices from G are present in H and adjacent in G, then they are also adjacent in H). The only vertices not coloured yet in G are the vertices of degree two that were suppressed. But since these have only two neighbours and $k \geq 3$ colours in their lists, they can be coloured without any problems.

(c) Give an example showing that we cannot take $k = 2$ in part (b).

Solution: All n-cycles with n even are 2-choosable, while those with n odd are not 2-choosable (question in the homework). So if we take H a 4-cycle and G a 5-cycle, then H can be obtained from G by a suppressing one vertex of degree two. But these two graphs fail the implication "H k-choosable $\Rightarrow$ G k-choosable".

Question 2

Theorem 7 from Lecture 4 reads as follows: "$1/n$ *is a threshold for* $G(n,p)$ *to contain a triangle.*" Generalise this theorem to give a threshold for $G(n,p)$ to contain a clique K_r, and give a proof.

Solution: Let $X = \sum_S X_S$ be the number of copies of K_r in $G(n,p$, where the sum is taken over all subsets of the vertex set of $G(n,p)$ of size r, and X_S is the indicator random variable that the set S induces a clique. We have $\mathbb{E}X = p^{\binom{r}{2}}\binom{n}{r}$ and $\mathbb{E}X^2 = \sum_{S,S'} \mathbb{E}X_S x_{S'}$. As in the lectures, $\mathrm{Var}X = \sum_{S,S'}(\mathbb{E}X_S X_{S'} - \mathbb{E}X_S \mathbb{E}X_{S'}) = \sum_{S,S'} \mathbb{E}X_S X_{S'} - p^{2\binom{r}{2}}$.

For S, S' sharing at most one vertex we have $\sum_{S,S'} \mathbb{E}X_S X_{S'} = p^{2\binom{r}{2}}$. For S, S' sharing i vertices we have $\sum_{S,S'} \mathbb{E}X_S X_{S'} = p^{2\binom{r}{2}-\binom{i}{2}}$.

Therefor we can calculate

$$\begin{aligned}
\mathrm{Var} X &= \sum_{i=2}^{r} \#\{S, S' \mid |S \cap S'| = i\} \times \left(p^{2\binom{r}{2}-\binom{i}{2}} - p^{2\binom{r}{2}}\right) \\
&= \sum_{i=2}^{r} \Theta_r(n^{2r-i}) \times \left(p^{2\binom{r}{2}-\binom{i}{2}} - p^{2\binom{r}{2}}\right) \\
&\le \sum_{i=2}^{r} \Theta_r(n^{2r-i}) \times p^{2\binom{r}{2}-\binom{i}{2}}.
\end{aligned}$$

We claim that the threshold is $p = n^{-2/(r-1)}$.

Indeed, suppose that $p \ll n^{-2/(r-1)}$. Then $\mathbb{E}X \le p^{\binom{r}{2}} n^r \to 0$ as $n \to 0$, so $\mathbb{P}(X \ge 1) \le \mathbb{E}X \to 0$, by Markow's inequality.

If $p \gg n^{-2/(r-1)}$, we have by Chebyshev's inequality that $\mathbb{P}(X = 0) \le \dfrac{\mathrm{Var} X}{(\mathbb{E}X)^2}$. The squared mean can be crudely lower-bounded by $p^{r(r-1)}(n/2r)^{2r}$, so we get

$$\mathbb{P}(X = 0) \le \sum_{i=2}^{r} \Theta_r(n^{-i}) \times p^{-\binom{i}{2}} = \sum_{i=2}^{r} \Theta_r\left(\frac{p^{-(i-1)/2}}{n}\right)^i \le \sum_{i=2}^{r} \Theta_r\left(\frac{p^{-(r-1)/2}}{n}\right)^i.$$

The condition of p is equivalent to $p^{-(r-1)/2} \to 0$, so $\mathbb{P}(X = 0) \to 0$, as required.

Question 3

Use the regularity lemma to prove that for any $\gamma, \nu > 0$ there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that if a graph G on $n \ge n_0$ vertices has minimum degree $(1/2 + \gamma)n$ and every set of vertices of size ηn contains an edge, then G contains a collection of vertex-disjoint triangles covering at least $(1 - \nu)n$ vertices of G.

To do this, execute the following steps in order.

1. Apply the regularity lemma with parameters $\varepsilon < \gamma^{1000} \times \nu^{1000}$ (to be read: ridiculously small compared to any reasonable function of both γ and ν) and $k_0 = 1/\varepsilon$. Consider the cluster graph $R(G)$ as defined in the lectures, except only take edges of weight at least $\gamma/4$.

   Show that this cluster graph has minimum degree at least $|R(G)|/2$.

2. Is there a result that guarantees a perfect matching in $R(G)$? Can we reduce the problem of finding vertex-disjoint triangles to a single regular pair (V_i, V_j)?

3. Show that if (U, V) is an ε-regular pair from the partition of density at least $\gamma/4$, and every set of ηn vertices contains an edge, then if η is sufficiently small compared to ε, we can carry out the following procedure iteratively, covering all but $2\varepsilon|U|$ and $2\varepsilon|V|$ vertices respectively on each side: Pick $v \in U$ of maximum degree to V; find an edge in its neighbourhood in V; remove the resulting triangle; switch sides.

Solution :

1. Let m be the size of a cluster of the regularity partition (except the junk set V_0 which has size at most εn), and let $k \leq K(\varepsilon)$ be the number of clusters. If there is a vertex $i \in V(R(G))$ of degree less than $|R(G)|/2$, then by definition the edges between the cluster V_i and the rest of the graph are at most

$$\frac{k}{2} \cdot \frac{\gamma}{4} \cdot m^2 + m \cdot \varepsilon n + \big(k - \frac{k}{2}\big) \cdot m^2.$$

(The first term counts the edges to sets V_j such that $ij \notin E(R(G))$; the second term counts the edges to V_0; and the third term counts the edges to sets V_j such that $ij \in E(R(G))$.)
Now we show that this somehow contradicts the minimum degree of G. By the minimum degree we have $e(V_i, G) \geq m(1/2 + \gamma)n \geq (1/2 + \gamma)km^2$ (edges within V_i are counted twice). Also, $m \cdot \varepsilon n \leq \frac{\gamma}{8} km^2$, since $km = n - |V_0| \geq (1 - \varepsilon)n$ and also $\varepsilon < \gamma^{1000}$. Simplifying the upper and lower bound we have

$$(1/2 + \gamma)km^2 \leq (1/2 + \gamma/4)km^2,$$

which is a contradiction. Therefore the k-vertex graph $R(G)$ has minimum degree at least $k/2$.

2. There is! It's Dirac's theorem which says that a k-vertex graph with degree at least $k/2$ has a Hamilton cycle. The cycle contains a perfect matching if k is even, or a matching on $k-1$ vertices if k is odd. Applying this to $R(G)$, we get a matching of $\lfloor k/2 \rfloor$ pairs of clusters. If we cover all but $\nu/2$ proportion of the vertices in each of these pairs with triangles, we will have covered all but $k\nu m/2 + m + \varepsilon n \leq \nu n$ vertices of G, so the problem reduces to covering all but $\nu/2$ proportion of the vertices of an ε-regular pair of density $\gamma/4$ with triangles. Note that it suffices to cover all but 2ε-proportion of vertices, as $2\varepsilon < \nu/2$.

3. Carry out the procedure until it is no longer possible. Let U', V' be the uncovered vertices, and suppose U' has size at least $2\varepsilon|U|$. The procedure was carried out in a balanced way, so V' must have size at least $\varepsilon|V|$. If there is a vertex $u \in U'$ with more than ηn neighbours in V', then the neighbourhood of u in V' must contain an edge (recall that η is much smaller than ε), which would form a triangle with u, contradicting the maximality of the set of vertex-disjoint triangles we consider. Therefore, $e(U', V') \leq |U'|\eta n \leq (\gamma/8)|U||V| \leq (\gamma/4 - \varepsilon)|U||V|$, by our choice of constants (ε much smaller than ν and γ, and $|V| \geq (1 - \varepsilon)n/K(\varepsilon) \geq 1000\eta n/\gamma$). By regularity this means that either U' has size less than $\varepsilon|U|$ or V' has size less than $\varepsilon|V|$, a contradiction.

END OF PAPER