LTCC Homological algebra mock exam

Let A be an algebra over a commutative ring k.

1. Let X be a chain complex of A-modules such that $H_n(X) = \{0\}$ for all but finitely many integers n.

(a) Show that there exist a bounded below complex of A-modules Y and a quasi-isomorphism $f : Y \to X$.
(*Hint:* use the fact that there is an integer r such that $H_n(X) = \{0\}$ for $n \leq r$, and define $Y_n = X_n$ for $n \geq r$, $Y_{r-1} = \mathrm{Im}(\delta_r)$, where δ is the differential of X, and $Y_n = \{0\}$ for $n < r - 1$).

(b) Show that there exist a bounded above complex of A-modules Z and a quasi-isomorphism $g : X \to Z$.

2. Let X be a chain complex of A-modules. For any integer n set $C(X)_n = X_{n-1} \oplus X_n$ and denote by $\Delta : C(X)_n \to C(X)_{n-1}$ the A-homomorphism given by

$$\Delta_n(x, y) = (-\delta_{n-1}(x), x + \delta_n(y))$$

(a) Show that $(C(X)_n, \Delta_n)_{n\in\mathbb{Z}}$ is a contractible chain complex.

(b) Show that there is an exact sequence of chain complexes

$$0 \longrightarrow X \longrightarrow C(X) \longrightarrow X[1] \longrightarrow 0$$

3. For any integer $n \geq 0$ calculate the abelian groups $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ and $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$.

4. Let $d : A \to A$ be a derivation. Show that $d(1) = 0$.

5. Suppose that k is a field. Calculate the dimension of $HH^1(k[x]/(x^2))$.
(*Hint:* calculate all derivations on $k[x]/(x^2)$.)

6. Show that for any A-module, the chain complex

$$\cdots \longrightarrow 0 \longrightarrow U \xrightarrow{\mathrm{Id}_U} U \longrightarrow 0 \longrightarrow \cdots$$

is contractible, where U is in any two consecutive degrees of the chain complex.