A course in homological algebra

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Introduction

Homological algebra emerged from algebraic topology as a tool to translate geometric intuition into algebraic formalism. Homological algebra shifts the focus from a direct descriptive approach of a mathematical object X in some category $\mathcal{C}$ towards a study of that object in relation to all others. One way to do this is to replace X by the covariant functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ or the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$. Homological algebra goes a step further by first embedding $\mathcal{C}$ in a larger category $\mathcal{D}$ endowed with a self-equivalence

$$\Sigma : \mathcal{D} \to \mathcal{D} \ .$$

This leads to considering the sets of morphisms in $\mathcal{D}$ of the form

$$\mathrm{Hom}_{\mathcal{D}}(X, \Sigma^n(Y))$$

where X, Y are objects in $\mathcal{C}$, now regarded as objects in the category $\mathcal{D}$, and where $n \in \mathbb{Z}$. If $\mathcal{C}$ is an abelian category with enough projective or injective objects, then the passage from $\mathcal{C}$ to $\mathcal{D}$ amounts typically to replacing an object in $\mathcal{C}$ by a projective or injective resolution, and $\mathcal{D}$ is what is known as a derived category, with Σ the shift functor. The new invariants thus obtained often describe obstructions regarding the existence and uniqueness of objects and maps between objects with certain properties.

Homological methods can be applied to a wide range of mathematical objects, and hence some basic category theoretic language is essential. The appendix contains a review of the terminology and background material from category theory which we will use from the start, and which can be found in many standard sources, such as [6], [8], [10].

The first three sections of this course cover three key concepts of homological algebra, namely complexes, homology, and homotopy. The focus in these sections is on methods how to manipulate chain complexes and their invariants. We will then describe how chain complexes are attached to a variety of mathematical objects, including algebras, groups, and topological spaces, and the interaction between the structure of these objects and the homological invariants attached to them.

Contents

Chapter 1

Complexes, homology, and homotopy

1.1 Complexes

Definition 1.1.1. A *graded object over a category* $\mathcal{C}$ is a family $X = (X_n)_{n\in\mathbb{Z}}$ of objects X_n in $\mathcal{C}$. Given an integer m and two graded objects $X = (X_n)_{n\in\mathbb{Z}}$ and $Y = (Y_n)_{n\in\mathbb{Z}}$ over $\mathcal{C}$, a *graded morphism of degree* m from X to Y is a family $f = (f_n)_{n\in\mathbb{Z}}$ of morphisms $f_n : X_n \to Y_{n+m}$ in $\mathcal{C}$. The category of graded objects over $\mathcal{C}$ with graded morphisms of degree zero is denoted $\mathrm{Gr}(\mathcal{C})$.

The notion of a graded object makes sense for any category. We adopt the convention that a graded object is a family of objects indexed by $\mathbb{Z}$, but it is worth noting that here may be situations where it is useful to specify gradings indexed by $\mathbb{N}$ or more general groups and monoids.

For instance, a graded vector space over a field k is an object in the category $\mathrm{Gr}(\mathbf{Vect}(k))$, and a graded A-module is a graded object over the category $\mathrm{Mod}(A)$ of (left unital) A-modules, where A is a ring or an algebra over a commutative ring. In an additive category in which direct sums indexed by $\mathbb{Z}$ exist, such as in the categories $\mathbf{Vect}(k)$ and $\mathrm{Mod}(A)$, one writes sometimes $\oplus_{n\in\mathbb{Z}}X_n$ instead of $(X_n)_{n\in\mathbb{Z}}$, where this notation is understood to include the structural monomorphisms $X_m \to \oplus_{n\in\mathbb{Z}}X_n$.

The minimum requirement for the following definition of a (co-) chain complex over a category $\mathcal{C}$ is the notion of a *zero morphism* between any two objects X and Y in $\mathcal{C}$. The existence of zero morphisms is ensured by the existence of a *zero object*. This is an object, typically denoted 0, which is *terminal* and *initial*; that is, there are unique morphisms $X \to 0$ and $0 \to X$ for all objects X in $\mathcal{C}$. The zero morphism $X \to Y$ is the unique morphism which factors $X \to 0 \to Y$. In practice $\mathcal{C}$ will usually be an *additive category*; that is, $\mathcal{C}$ has a zero object, any two objects in $\mathcal{C}$ have a direct sum, and the morphism spaces between objects are abelian groups such that the composition in $\mathcal{C}$ is $\mathbb{Z}$-bilinear. See 6.4.3 for more details.

Definition 1.1.2. A *chain complex* over a category $\mathcal{C}$ with a zero object is a pair (X, δ) consisting of a graded object X in $\mathcal{C}$ and a graded endomorphism δ of degree -1, called the *differential of the* *complex* , satisfying $\delta \circ \delta = 0$. Explicitly, δ is a family of morphisms $\delta_n : X_n \to X_{n-1}$ satisfying $\delta_{n-1} \circ \delta_n = 0$. Dually, a *cochain complex* over a $\mathcal{C}$ is a pair (X, δ) consisting of a graded object $X = (X^n)_{n\in\mathbb{Z}}$ in $\mathcal{C}$ and a graded endomorphism $\delta = (\delta^n : X^n \to X^{n+1})_{n\in\mathbb{Z}}$, called *differential* of the cochain complex, of degree 1 satisfying $\delta \circ \delta = 0$, or equivalently, $\delta^{n+1} \circ \delta^n = 0$ for $n \in \mathbb{Z}$.

One can visualise a chain complex as an infinite sequence of morphisms in which the composition of any two consecutive morphisms is zero.

$$\cdots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} \cdots$$

For cochain complexes, the only difference is that the indices increase in the direction of the differential:

$$\cdots \longrightarrow X^{n-1} \xrightarrow{\delta^{n-1}} X^n \xrightarrow{\delta^n} X_{n+1} \xrightarrow{\delta^{n+1}} \cdots$$

In order to distinguish between chain complexes and cochain complexes, the standard notational convention is to use subscripts for chain complexes and superscripts for cochain complexes. One can always switch from a chain complex to a cochain complex and vice versa by setting $X^n = X_{-n}$ and $\delta^n = \delta_{-n}$. Through this correspondence, any terminology in the context of chain complexes has an analogue for cochain complexes.

Example 1.1.3. Let $\mathcal{C}$ be a category with a zero object. Let U, V be objects in $\mathcal{C}$. Any morphism $f : U \to V$ in $\mathcal{C}$ can be regarded as a chain or cochain complex of the form

$$\cdots \longrightarrow 0 \longrightarrow U \xrightarrow{f} V \longrightarrow 0 \longrightarrow \cdots$$

with U and V in any two consecutive degrees. A special case of this example arises for A an algebra and c any element in A. Then the map $f : A \to A$ given by $f(a) = ac$ for all $a \in A$ (that is, f is given by right multiplication with c) is an A-module endomorphism, giving rise to a two-term complex of the form

$$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{f} A \longrightarrow 0 \longrightarrow \cdots$$

Tensor products (to be defined) of complexes of this form yield Koszul complexes.

Definition 1.1.4. Let $\mathcal{C}$ be a category with a zero object. A *chain map* between two chain complexes (X, δ), (Y, ϵ) over $\mathcal{C}$ is a graded morphism of degree zero $f = (f_n : X_n \to Y_n)_{n\in\mathbb{Z}}$ satisfying

$$f \circ \delta = \epsilon \circ f \ ,$$

or equivalently,

$$f_{n-1} \circ \delta_n = \epsilon_n \circ f_n$$

for all $n \in \mathbb{Z}$. Cochain maps are defined similarly. The chain complexes, together with chain maps, form the category $\mathrm{Ch}(\mathcal{C})$ of *chain complexes over* $\mathcal{C}$.

A chain map can be visualised as a commutative ladder of the form

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & X_{n+1} & \xrightarrow{\delta_{n+1}} & X_n & \xrightarrow{\delta_n} & X_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots \\
& {\scriptstyle f_{n+1}}\downarrow & & {\scriptstyle f_n}\downarrow & & {\scriptstyle f_{n-1}}\downarrow & & \\
\cdots \longrightarrow & Y_{n+1} & \xrightarrow{\epsilon_{n+1}} & Y_n & \xrightarrow{\epsilon_n} & Y_{n-1} & \xrightarrow{\epsilon_{n-1}} & \cdots
\end{array}$$

If the differential of a complex (X, δ) is clear from the context, we adopt the notational abuse of just calling X a chain complex. We have a forgetful functor $\mathrm{Ch}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$ mapping a chain complex (X, δ) to its underlying graded object X. We have a functor from $\mathrm{Gr}(\mathcal{C}) \to \mathrm{Ch}(\mathcal{C})$ sending a graded object X to the complex $(X, 0)$ with zero differential; when composed with the forgetful functor this yields the identity functor on $\mathrm{Gr}(\mathcal{C})$. Note that complexes and chain maps can be defined in any category with a zero object.

For the next concept we need slightly more structure, namely that $\mathcal{C}$ is an additive category (so that in particular morphism sets have an abelian group structure). For any integer i, we define the *shift automorphism* $[i]$ of $\mathrm{Ch}(\mathcal{C})$ as follows. For (X, δ) a chain complex, we define a graded object $X[i]$ by setting

$$X[i]_n = X_{n-i} \ ;$$

this becomes a chain complex together with the differential $\delta[i]$ defined by

$$\delta[i]_n = (-1)^i \delta_{n-i} \ ,$$

for $n \in \mathbb{Z}$. Note the sign convention here (this is where we need $\mathcal{C}$ to be additive, so that morphism sets between objects are abelian groups; the minus sign amounts to taking the inverse with respect to the abelian group structure of the relevant morphism space). This defines the shift functor on graded objects and on chain complexes. We need to define the shift functor on chain maps. For $f : X \to Y$ a chain map, we define $f[i] : X[i] \to Y[i]$ by setting

$$f[i]_n = f_{n-1}$$

for all $n \in \mathbb{Z}$. We define a shift functor on the category of cochain complexes analogously; that is, for (X, δ) a cochain complex, we define $(X[i], \delta[i])$ by $X[i]^n = X^{n+i}$ and $\delta[i]^n = (-1)^i \delta^{n+i}$ for all $n \in \mathbb{Z}$.

A chain complex X is called *bounded above* if $X_n = 0$ for n large enough; we denote by $\mathrm{Ch}^+(\mathcal{C})$ the full subcategory of $\mathrm{Ch}(\mathcal{C})$ consisting of all bounded above chain complexes over $\mathcal{C}$. A chain complex X is called *bounded below* if $X_n = 0$ for n small enough; we denote by $\mathrm{Ch}^-(\mathcal{C})$ the full subcategory of $\mathrm{Ch}(\mathcal{C})$ consisting of all bounded below chain complexes over $\mathcal{C}$. A chain complex X is called *bounded* if $X_n = 0$ for all but finitely many i; we denote by $\mathrm{Ch}^b(\mathcal{C})$ the full subcategory of $\mathrm{Ch}(\mathcal{C})$ consisting of all bounded chain complexes over $\mathcal{C}$.

Exercise 1.1.5. Let $\mathcal{C}$ be a category, and let X, Y, Z be graded objects over $\mathcal{C}$. Let $f : X \to Y$ be a graded map of degree i, and let $g : Y \to Z$ be a graded map of degree j. Show that $g \circ f$ is a graded map of degree $i + j$.

Exercise 1.1.6. Let $\mathcal{C}$ be an additive category, and let X, Y be bounded chain complexes over $\mathcal{C}$. Show that there are only finitely many integers i such that the space $\mathrm{Hom}_{\mathcal{H}(\mathcal{C})}(X, Y[i])$ of chain maps from X to $Y[i]$ is nonzero.

Exercise 1.1.7. Let $\mathcal{C}$ be an additive category, let i, $j \in \mathbb{Z}$. Show that for any chain complex X over $\mathcal{C}$ we have $X[i + j] = (X[i])[j]$. Show that this is an equality of functors $[i + j] = [j] \circ [i]$. Deduce that $[i]$ and $[-i]$ are inverse functors.

Exercise 1.1.8. Show that if $\mathcal{C}$ is an abelian category (e. g. the module category $\mathrm{Mod}(A)$ of an algebra A), then $\mathrm{Ch}(\mathcal{C})$ is again an abelian category. More precisely, show that for any chain map f from a complex (X, δ) to a complex (Y, ϵ), the differential δ restricts to a differential on the graded object $\ker(f) = (\ker(f_n : X_n \to Y_n))_{n\in\mathbb{Z}}$, and the resulting chain complex $(\ker(f), \delta|_{\ker(f)})$ is a kernel of f; similarly, show that ϵ induces a differential on the cokernel of f as graded morphism, which yields a cokernel of f in the category $\mathrm{Ch}(\mathcal{C})$. Deduce that f is a monomorphism (resp. epimorphism) in $\mathrm{Ch}(\mathcal{C})$ if and only if all f_i are monomorphisms (resp. epimorphisms) in $\mathcal{C}$. Show finally that the categories $\mathrm{Ch}^+(\mathcal{C})$, $\mathrm{Ch}^-(\mathcal{C})$, and $\mathrm{Ch}^b(\mathcal{C})$ are full abelian subcategories of $\mathrm{Ch}(\mathcal{C})$.

1.2 Homology

The content of this section could be formulated for arbitrary abelian categories. For expository purpose, we consider module categories, keeping in mind that this is no loss of generality, thanks to the Freyd-Mitchell embedding theorem, which says that any small abelian category is equivalent to a full subcategory of a module category. For a sketch of a proof of this theorem and further references, see Weibel [12, §1.6].

Throughout this section, A is an algebra over a commutative ring k. For a complex (X, δ) of A-modules, the condition $\delta \circ \delta = 0$ means that we have an inclusion $\mathrm{Im}(\delta) \subseteq \ker(\delta)$. The cokernel $\ker(\delta)/\mathrm{Im}(\delta)$ of this inclusion is defined to be the homology of X.

Definition 1.2.1. The *homology* of a chain complex (X, δ) of A-modules is the graded A-module $H_*(X, \delta) = \ker(\delta)/\mathrm{Im}(\delta)$; more explicitly,

$$H_n(X, \delta) = \ker(\delta_n)/\mathrm{Im}(\delta_{n+1})$$

for $n \in \mathbb{Z}$. If the differential δ is clear from the context we write $H_*(X)$ instead of $H_*(X, \delta)$. If $H_*(X) = \{0\}$ then X is called *exact* or *acyclic* . Similarly, the *cohomology* of a cochain complex (Y, ϵ) of A-modules is the graded A-module $\ker(\epsilon)/\mathrm{Im}(\epsilon)$; explicitly,

$$H^n(Y, \epsilon) = \ker(\epsilon^n)/\mathrm{Im}(\epsilon^{n-1})$$

for $n \in \mathbb{Z}$. As before, if ϵ is clear from the context we write $H^*(Y)$, and if $H^*(Y) = \{0\}$ then Y is called *exact* or *acyclic*.

The homology $H_n(X)$ in a fixed degree n of a chain complex X is a subquotient of X_n.

Exercise 1.2.2. Let (X, δ) be a chain complex of A-modules. Show that if $\delta = 0$, then $H_*(X, \delta) = X$, or equivalently, $H_n(X, \delta) = X_n$ for all $n \in \mathbb{Z}$.

Example 1.2.3. Any A-module M can be viewed as a complex 'concentrated in degree 0' by setting $X_0 = M$ and $X_n = \{0\}$ for any nonzero integer n, with the zero differential. The homology of this complex is isomorphic to the graded object X because the differential is zero.

Example 1.2.4. Any short exact sequence of A-modules can be viewed as bounded exact complex.

Taking homology or cohomology is functorial:

Proposition 1.2.5. *Let $f : (X, \delta) \to (Y, \epsilon)$ be a chain map of chain complexes of A-modules. Then f restricts to graded maps $\ker(\delta) \to \ker(\epsilon)$ and $\mathrm{Im}(\delta) \to \mathrm{Im}(\epsilon)$. In particular, f induces a graded homomorphism of graded A-modules $H_*(f) : H_*(X) \to H_*(Y)$. The assignments $X \mapsto H_*(X)$ and $f \mapsto H_*(f)$ define a functor from the category of chain complexes $\mathrm{Ch}(\mathrm{Mod}(A))$ to the category of graded A-modules $\mathrm{Gr}(\mathrm{Mod}(A))$.*

Proof. Let n be an integer. Let $x \in \ker(\delta_n)$. We have $\epsilon_n(f_n(x)) = f_{n-1}(\delta_n(x)) = 0$, hence $f(x) \in \ker(\epsilon_n)$. Thus f sends $\ker(\delta)$ to $\ker(\epsilon)$. Let $y \in \mathrm{Im}(\delta_{n+1})$. Write $y = \delta_{n+1}(z)$ for some $z \in X_{n+1}$. We have $f_n(y) = f_n(\delta_{n+1}(z)) = \epsilon_{n+1}(f_{n+1}(z)) \in \mathrm{Im}(\epsilon_{n+1})$, and hence f sends $\mathrm{Im}(\delta)$ to $\mathrm{Im}(\epsilon)$. Since $H_*(f)$ is induced by f, one verifies easily that given two composable chain maps f, g, we have $H_*(g \circ f) = H^*(g) \circ H^*(f)$. □

We have the obvious analogous statement for cohomology. The functor induced by taking homology is very different from the forgetful functor $\mathrm{Ch}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$.

Definition 1.2.6. Let $\mathcal{C}$ be an abelian category and X, Y chain complexes. A chain map $f : X \to Y$ is called a *quasi-isomorphism* if the induced map on homology $H_*(f)$ is an isomorphism $H_*(X) \cong H_*(Y)$ in $\mathrm{Gr}(\mathcal{C})$.

Note that if $f : X \to Y$ is a quasi-isomorphism, there need not be a chain map $g : Y \to X$ inducing the inverse isomorphism $H_*(Y) \cong H_*(X)$. The relation of two complexes X, Y being quasi-isomorphic is the smallest equivalence relation $\sim$ satisfying $X \sim Y$ if there is a quasi-isomorphism $X \to Y$. Similarly for cochain maps between cochain complexes.

Example 1.2.7. Let $0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$ be a short exact sequence of A-modules; that is, f is injective, $\mathrm{Im}(f) = \ker(g)$ and g is surjective. Then in particular $g \circ f = 0$, and hence the following diagram is commutative:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & U & \xrightarrow{f} & V & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & W & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

is commutative. This diagram represents a chain map from the top row to the bottom row, viewed as chain complexes with V, W in degree zero, with U in degree 1, and all other terms zero. This chain map is a quasi-isomorphism: the homology of the top row in degree zero is $V/\mathrm{Im}(f) = V/\ker(g) \cong \mathrm{Im}(g) = W$. In all other degrees the homology of the top row is zero (using that f is injective). The homology of the bottom row is also W in degree 0 and zero in all other degrees. Thus g induces a quasi-isomorphism. In general, there is no quasi-isomorphism from the bottom to the top row. In fact, there is such a quasi-isomorphism if and only if the exact sequence we started out with is split. To see this, observe first that any A-homomorphism $s : W \to V$ determines a chain map from the bottom to the top row (which is zero in all nonzero degrees). Such an s is a quasi-isomorphism if and only if the composition of maps $W \xrightarrow{s} V \longrightarrow V/\mathrm{Im}(f)$ is an isomorphism. This composition is injective if and only of s is injective and $\mathrm{Im}(s) \cap \mathrm{Im}(f) = \{0\}$. This composition is surjective if and only if $\mathrm{Im}(f) + \mathrm{Im}(s) = V$. Thus this is an isomorphism if and only if s is injective and if $V = \mathrm{Im}(s) \oplus \mathrm{Im}(f)$, so if and only if g is split surjective.

One might hope that if X is a subcomplex of a chain complex Y, then the homology of Y can be calculated in terms of that of X and Z. It is not quite as simple as that. One of the fundamental features of complexes over an abelian category is that short exact sequences of complexes give rise to long exact (co-)homology sequences. We state and prove this for module categories, as this will be sufficient for this course. For general abelian categories, one can either directly modify the proofs, or use Freyd's embedding theorem, saying that any abelian category can be fully embedded into a module category.

Theorem 1.2.8. *Let A be a k-algebra. Any short exact sequence of chain complexes of A-modules*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{H_n(g)} H_n(Z) \xrightarrow{d_n} H_{n-1}(X) \longrightarrow \cdots$$

depending functorially on the short exact sequence.

The functorial dependence in this theorem means that given a commutative diagram of chain complexes with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \longrightarrow & 0
\end{array}$$

we get a commutative 'ladder' of long exact sequences

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & H_n(X) & \xrightarrow{H_n(f)} & H_n(Y) & \xrightarrow{H_n(g)} & H_n(Z) & \xrightarrow{d_n} & H_{n-1}(X) & \xrightarrow{H_{n-1}(f)} & H_{n-1}(Y) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle H_n(a)} & & \downarrow{\scriptstyle H_n(b)} & & \downarrow{\scriptstyle H_n(c)} & & \downarrow{\scriptstyle H_{n-1}(a)} & & \downarrow{\scriptstyle H_{n-1}(b)} & & \\
\cdots & \longrightarrow & H_n(X') & \xrightarrow[H_n(f')]{} & H_n(Y') & \xrightarrow[H_n(g')]{} & H_n(Z') & \xrightarrow[d'_n]{} & H_{n-1}(X') & \xrightarrow[H_{n-1}(f')]{} & H_{n-1}(Y) & \longrightarrow & \cdots
\end{array}$$

The morphism d_n is called *connecting homomorphism.* Theorem 1.2.8 translates verbatim to cochain complexes, except that the connecting homomorphism $d^n : H^n(Z) \to H^{n+1}(X)$ is of degree 1.

Proof of Theorem 1.2.8. Denote by δ, ϵ, ζ the differentials of X, Y, Z, respectively. The short exact sequence in the statement is a commutative diagram of the following form.

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} & \xrightarrow{g_{n+1}} & Z_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_{n+1}} & & \downarrow{\scriptstyle \epsilon_{n+1}} & & \downarrow{\scriptstyle \zeta_{n+1}} & & \\
0 & \longrightarrow & X_n & \xrightarrow{f_n} & Y_n & \xrightarrow{g_n} & Z_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \epsilon_n} & & \downarrow{\scriptstyle \zeta_n} & & \\
0 & \longrightarrow & X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Z_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_{n-1}} & & \downarrow{\scriptstyle \epsilon_{n-1}} & & \downarrow{\scriptstyle \zeta_{n-1}} & & \\
0 & \longrightarrow & X_{n-2} & \xrightarrow{f_{n-2}} & Y_{n-2} & \xrightarrow{g_{n-2}} & Z_{n-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

where the horizontal sequences are exact; that is, f_n is injective, g_n is surjective, and $\mathrm{Im}(f_n) = \ker(g_n)$ for all $n \in \mathbb{Z}$.

We define $d_n : H_n(Z) \to H_{n-1}(X)$ as follows. By the very definition of $H_n(Z) = \ker(\zeta_n)/\mathrm{Im}(\zeta_{n+1})$, any element in $H_n(Z)$ is of the form

$$z + \mathrm{Im}(\zeta_{n+1})$$

for some $z \in \ker(\zeta_n)$. Since g is surjective in each degree, there is $y \in Y_n$ such that

$$g_n(y) = z \ .$$

Since $\zeta_n(z) = 0$ we get that

$$g_{n-1}(\epsilon_n(y)) = \zeta_n(g_n(y)) = \zeta_n(z) = 0 \ .$$

Thus $\epsilon_n(y) \in \ker(g_{n-1}) = \mathrm{Im}(f_{n-1})$, where we use the exactness of the sequence in the statement. Thus there is $x \in X_{n-1}$ satisfying $f_{n-1}(x) = \epsilon_n(y)$. Moreover,

$$f_{n-2}(\delta_{n-1}(x)) = \epsilon_{n-1}(f_{n-1}(x)) = \epsilon_{n-1}(\epsilon_n(y)) = 0 \ .$$

Since f_{n-2} is a monomorphism, this shows that $\delta_{n-1}(x) = 0$, or equivalently, we have

$$x \in \ker(\delta_{n-1}) \ .$$

Hence $x + \mathrm{Im}(\delta_n)$ is an element in $H_{n-1}(X)$. In order to show that the assignment

$$z + \mathrm{Im}(\zeta_{n+1}) \mapsto x + \mathrm{Im}(\delta_n)$$

yields a well-defined map one needs to verify that if $z \in \mathrm{Im}(\zeta_{n+1})$ then $x \in \mathrm{Im}(\delta_n)$. Suppose that $z \in \mathrm{Im}(\zeta_{n+1})$. Write $z = \zeta_{n+1}(s)$ for some $s \in Z_{n+1}$. Since g_{n+1} is surjective there is $t \in Y_{n+1}$ such that $g_{n+1}(t) = s$. Then

$$g_n(\epsilon_{n+1}(t)) = \zeta_{n+1}(g_{n+1}(t)) = \zeta_{n+1}(s) = z = g_n(y)$$

and hence $y - \epsilon_{n+1}(t) \in \ker(g_n) = \mathrm{Im}(f_n)$. Write $y - \epsilon_{n+1}(t) = f_n(w)$ for some $w \in X_n$. We have

$$f_{n-1}(\delta_n(w)) = \epsilon_n(f_n(w)) = \epsilon_n(y - \epsilon_{n+1}(t)) = \epsilon_n(y) = f_{n-1}(x)$$

Since f_{n-1} is injective, this implies that $x = \delta_n(w)$ belongs to $\mathrm{Im}(\delta_n)$. This implies that there is indeed a well-defined map $d_n : H_n(Z) \to H_{n-1}(X)$ such that

$$d_n(z + \mathrm{Im}(\zeta_{n+1})) = x + \mathrm{Im}(\delta_n)$$

with x and z as above.

We need to show the exactness of the the sequence in three places. We start with showing $\mathrm{Im}(H_n(f)) = \ker(H_n(g))$. We have

$$\mathrm{Im}(H_n(f)) = \{f_n(x) + \mathrm{Im}(\epsilon_{n+1}) \mid x \in \ker(\delta_n)\}$$

$$\ker(H_n(g)) = \{y + \mathrm{Im}(\epsilon_{n+1}) \mid y \in \ker(\epsilon_n),\ g_n(y) \in \mathrm{Im}(\zeta_{n+1})\}$$

The inclusion $\mathrm{Im}(H_n(f)) \subseteq \ker(H_n(g))$ is clear because $g \circ f = 0$, hence $H_n(g) \circ H_n(f) = H_n(g \circ f) = 0$ by the functoriality of H_n. For the reverse inclusion, let $y \in \ker(\epsilon_n)$ such that $g_n(y) \in \mathrm{Im}(\zeta_{n+1})$. Thus $g_n(y) = \zeta_{n+1}(z')$ for some $z' \in Z_{n+1}$. Since g_{n+1} is surjective, there is $y' \in Y_{n+1}$ such that $g_{n+1}(y') = z'$. Then

$$g_n(y - \epsilon_{n+1}(y')) = g_n(y) - g_n(\epsilon_{n+1}(y')) = \zeta_{n+1}(z') - \zeta_{n+1}(g_{n+1}(y')) = \zeta_{n+1}(z') - \zeta_{n+1}(z') = 0$$

Thus $y - \epsilon_{n+1}(y') \in \ker(g_n) = \mathrm{Im}(f_n)$. Write $y - \epsilon_{n+1}(y') = f_n(x')$ for some $x' \in X_n$. Then $y + \mathrm{Im}(\epsilon_{n+1}) = f_n(x') + \mathrm{Im}(\epsilon_{n+1}) \in \mathrm{Im}(H_n(f))$.

We need to show next that $\mathrm{Im}(H_n(g)) = \ker(d_n)$. We have

$$\mathrm{Im}(H_n(g)) = \{g_n(y) + \mathrm{Im}(\zeta_{n+1}) \mid y \in \ker(\epsilon_n)\}$$

For $\ker(d_n)$ we need to determine all $z \in \ker(\zeta_n)$ such that the element $x \in \ker(\delta_{n-1})$ constructed above lies actually in $\mathrm{Im}(\delta_n)$, so that x represents 0 in $H_{n-1}(X)$. Write as before $z = g_n(y)$ for some $y \in Y_n$ and $\epsilon_n(y) = f_{n-1}(x)$ for some $x \in \ker(\delta_{n-1})$, so that $x + \mathrm{Im}(\delta_n) = d_n(z + \mathrm{Im}(\epsilon_{n+1})$ as described above. Suppose that $z + \mathrm{Im}(\epsilon_{n+1}) \in \mathrm{Im}(H_n(g))$. That is, we have $z + \mathrm{Im}(\epsilon_{n+1}) = g_n(y) + \mathrm{Im}(\epsilon_{n+1})$ for some $y \in \ker(\epsilon_n)$. Then $0 = \epsilon_n(y) = f_{n-1}(x)$, which shows that $x = 0$ as f_{n-1} is injective. This shows the inclusion $\mathrm{Im}(H_n(g)) \subseteq \ker(d_n)$. Suppose conversely that $z + \mathrm{Im}(\zeta_{n+1}) \in \ker(d_n)$. This is equivalent to $x \in \mathrm{Im}(\delta_n)$. Write $x = \delta_n(x')$ for some $x' \in X_n$. Set $y' = f_n(x')$. Then

$$\epsilon(y') = \epsilon(f_n(x')) = f_{n-1}(\delta_n(x')) = f_{n-1}(x)$$

and

$$g_n(y - y') = g_n(y) - g_n(f_n(x')) = g_n(y) = z$$

The above implies

$$\epsilon_n(y - y') = \epsilon_n(y) - \epsilon_n(y') = f_{n-1}(x) - f_{n-1}(x) = 0$$

and thus $z = g_n(y - y')$ and $y - y' \in \ker(\epsilon_n)$ which means exactly that the class of z is in the image of $H_n(g)$.

The last verification for the exactness is $\mathrm{Im}(d_n) = \ker(H_{n-1}(f))$. By the construction of d_n, $\mathrm{Im}(d_n)$ consists of all classes $x + \mathrm{Im}(\delta_n)$ such that $f_{n-1}(x) = \epsilon_{(}(y)$ for some $y \in Y_n$ satisfying $g_n(y) = z \in \ker(\zeta_n)$. We have

$$\ker(H_{n-1}(f)) = \{x + \mathrm{Im}(\delta_n) \mid x \in \ker(\delta_n),\ f_{n-1}(x) \in \mathrm{Im}(\epsilon_n)\}$$

The inclusion $Im(d_n) \subseteq \ker(H_{n-1}(f))$ is clear from this description. For the converse, suppose that $f_{n-1}(x) = \epsilon_n(y)$ for some $y \in Y_n$. Consider $z = g_n(y)$. We have

$$\zeta_n(z) = \zeta_n(g_n(y)) = g_{n-1}(\epsilon_n(y)) = g_{n-1}(f_{n-1}(x)) = 0$$

and hence $z \in \ker(\zeta_n)$, which shows the equality as required.

This concludes the proof of the exactness statement. It remains to verify the naturality of the connecting homomorphisms. Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of chain maps such that the rows are exact. We need to show that the ladder above is commutative. The squares which involve only H_n are commutative because H_n is a functor; that is, we have a commutative diagram

$$\begin{array}{ccccc} H_n(X) & \xrightarrow{H_n(f)} & H_n(Y) & \xrightarrow{H_n(g)} & H_n(Z) \\ \downarrow{\scriptstyle H_n(a)} & & \downarrow{\scriptstyle H_n(b)} & & \downarrow{\scriptstyle H_n(c)} \\ H_n(X') & \xrightarrow[H_n(f')]{} & H_n(Y') & \xrightarrow[H_n(g')]{} & H_n(Z') \end{array}$$

It remains to show the commutativity of the diagram

$$\begin{array}{ccc} H_n(Z) & \xrightarrow{d_n} & H_{n-1}(X) \\ \downarrow{\scriptstyle H_n(c)} & & \downarrow{\scriptstyle H_{n-1}(a)} \\ H_n(Z') & \xrightarrow[d'_n]{} & H_{n-1}(X') \end{array}$$

Let x, y, z are as above in the construction of d_n; that is, $z \in \ker(\zeta_n)$, $z = g_n(y)$, and $f_{n-1}(x) = \epsilon_n(y)$. We need to show that the images $a_{n-1}(x)$, $b_n(y)$, $c_n(z)$ of x, y, z are the corresponding elements required in the construction of d'_n evaluated at the class of $c_n(z)$. That is, we need to verify that

$$g'_n(b_n(y)) = c_n(z)$$

and that

$$f'_{n-1}(a_{n-1}(x)) = \epsilon'_n(b_n(y))$$

The first equation holds because $g'_n(b_n(y)) = c_n(g_n(y)) = c_n(x)$. The second equation holds because $f'_{n-1}(a_{n-1}(x)) = b_{n-1}(f_{n-1}(x)) = b_{n-1}(\epsilon_n(y)) = \epsilon'_n(b_n(y))$. This shows the commutativity in the ladder above for the square involving d_n, completing the proof. □

Corollary 1.2.9. *Let A be a k-algebra and let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of chain complexes of A-modules.

(i) f is a quasi-isomorphism if and only if Z is acyclic.

(ii) g is a quasi-isomorphism if and only if X is acyclic.

(iii) If two of the complexes X, Y, Z are acyclic, so is the third.

Proof. The long exact homology sequence shows that if $H_{n+1}(Z) = H_n(Z) = \{0\}$, then $H_n(f)$ is an isomorphism, and if $H_n(f)$, $H_{n-1}(f)$ are isomorphisms, then the maps $H_n(g)$, d_n are zero, hence $H_n(Z) = \{0\}$. This shows (i), and the rest follows similarly. □

The following observation is used to compare the long exact homology sequences via a commutative ladder as above:

Proposition 1.2.10 (The 5-Lemma)**.** *Let A be a k-algebra and let*

$$\begin{array}{ccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & X_5 \\ \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & \downarrow{\scriptstyle a_4} & & \downarrow{\scriptstyle a_5} \\ Y_1 & \xrightarrow[g_1]{} & Y_2 & \xrightarrow[g_2]{} & Y_3 & \xrightarrow[g_3]{} & Y_4 & \xrightarrow[g_4]{} & Y_5 \end{array}$$

be a commutative diagram of A-modules with exact rows. If a_1, a_2, a_4, a_5 are isomorphisms then a_3 is an isomorphism.

Proof. Let $x \in \ker(a_3)$. Then $a_4(f_3(x)) = g_3(a_3(x)) = 0$, hence $f_3(x) = 0$ as a_4 is an isomorphism. Thus $x \in \ker(f_3) = \mathrm{Im}(f_2)$, and so there is $y \in X_2$ such that $f_2(y) = x$. Then $g_2(a_2(y)) = a_3(f_2(y)) = a_3(x) = 0$, hence $a_2(y) \in \ker(g_2) = \mathrm{Im}(g_1)$, and so there is $z \in Y_1$ satisfying $g_1(z) = a_2(y)$. As a_1 is an isomorphism, there is $w \in X_1$ such that $a_1(w) = z$. Then $a_2(f_1(w)) = g_1(a_1(w)) = g_1(z) = a_2(y)$. Since a_2 is an isomorphism this implies that $f_1(w) = y$. But then $x = f_2(y) = f_2(f_1(w)) = 0$, and so a_3 is injective. For the surjectivity of a_3, let $y \in Y_3$. Then $g_3(y) \in Y_4$. Since a_4 is an isomorphism, there is $v \in X_4$ such that $a_4(v) = g_3(y)$. Then $a_5(f_4(v)) = g_4(a_4(v)) = g_4(g_3(y)) = 0$. Thus $f_4(v) = 0$ as a_4 is an isomorphism. It follows that $v \in \ker(f_4) = \mathrm{Im}(f_3)$. Write $v = f_3(u)$ for some $u \in X_3$. Then $g_3(a_3(u) - y) = g_3(a_3(u)) - g_3(y) = a_4(f_3(u)) - g_3(y) = a_4(v) - g_3(y) = 0$. Thus $a_3(u) - y \in \ker(g_3) = \mathrm{Im}(g_2)$. Write $a_3(u) - y = g_2(w)$ for some $w \in y_2$. Since a_2 is an isomorphism there is $r \in X_2$ such that $a_2(r) = w$. Then $a_3(v) - y = g_2(w) = g_2(a_2(r)) = a_3(f_2(r))$. This shows that $y = a_3(v - f_2(r))$, and hence that a_3 is surjective. □

Corollary 1.2.11. *Let A be an algebra over a commutative ring k and let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of chain complexes of A-modules with exact rows. If two of a, b, c are quasi-isomorphisms, so is the third.

Proof. Apply the 5-Lemma to the five terms in the commutative ladder following Theorem 1.2.8. □

Exercise 1.2.12. State the cohomology version of Theorem 1.2.8 for cochain complexes.

Exercise 1.2.13. Describe the connecting homomorphism in the long exact cohomology sequence associated with a short exact sequence of cochain complexes.

Remark 1.2.14. Theorem 1.2.8 shows that every monomorphism and every epimorphism between chain complexes gives rise to a long exact homology sequence. One can use this to show that in fact *every* chain map $f : X \to Y$ between two chain complexes X, Y yields a long exact homology sequence. This is done by either adding a complex with zero homology to Y which allows us to replace f by a monomorphism without changing the homology of Y, or by adding a complex with zero homology to X which allows us to replace f by an epimorphism, without changing the homology of X. This can be done a in a canonical way, up to *homotopy*, the concept introduced in the next section. This leads to considering the notion of *triangulated categories*.

1.3 Homotopy

Definition 1.3.1. Let $\mathcal{C}$ be an additive category and let $(X\delta)$, (Y, ϵ) be complexes over $\mathcal{C}$. A *(chain) homotopy from* X *to* Y is a graded morphism $h : X \to Y$ of degree 1; that is, h is a family of morphisms $h_n : X_n \to Y_{n+1}$ in $\mathcal{C}$, for any $n \in \mathbb{Z}$. Two chain morphisms $f, f' : X \to Y$ are called *homotopic* , written $f \sim f'$, if there is a homotopy $h : X \to Y$ such that

$$f - f' = h \circ \delta + \epsilon \circ h \ ,$$

or equivalently, if

$$f_n - f'_n = h_{n-1} \circ \delta_n + \epsilon_{n+1} \circ h_n$$

for any $n \in \mathbb{Z}$.

Note that there is no requirement for a homotopy to be compatible with the differentials. For cochain complexes, we define analogously a *cochain homotopy* to be a graded morphism of degree -1 (so the degree of a homotopy is always opposite to that of the differential). One can visualise a homotopy between chain complexes by a diagram of the form

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{n+1} & \xrightarrow{\delta_{n+1}} & X_n & \xrightarrow{\delta_n} & X_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots \\
& \swarrow^{h_{n+1}} & & \swarrow^{h_n} & & \swarrow^{h_{n-1}} & & & \\
\cdots & \longrightarrow & Y_{n+1} & \xrightarrow{\epsilon_{n+1}} & Y_n & \xrightarrow{\epsilon_n} & Y_{n-1} & \xrightarrow{\epsilon_{n-1}} & \cdots
\end{array}$$

Definition 1.3.2. Let $\mathcal{C}$ be an additive category and let (X, δ), (Y, ϵ) be complexes over $\mathcal{C}$. A chain map $f : X \to Y$ is a *homotopy equivalence* if there is a chain map $g : Y \to X$ such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$; in that case, g is called a *homotopy inverse of* f, and the complexes X, Y are said to be *homotopy equivalent*, written $X \simeq Y$. If $X \simeq 0$ (the zero complex), then X is called *contractible*.

Proposition 1.3.3. *A chain complex X over an additive category $\mathcal{C}$ is contractible if and only if* $\mathrm{Id}_X \sim 0$ *(the zero chain map on X).*

Proof. We have $X \simeq 0$ if and only if there are maps $g : 0 \to X$ and $f : X \to 0$ such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_0$. The chain maps f, g, Id_0 are all zero because they start or end at the zero complex. Thus the equality $f \circ g = \mathrm{Id}_0$ holds trivially, and the statement $f \circ g \sim \mathrm{Id}_X$ is equivalent to $0 \sim \mathrm{Id}_X$, whence the result. □

The terminology comes from topology: complexes calculating the singular homology of a contractible topological space are contractible chain complexes.

Proposition 1.3.4. *Let X, Y be chain complexes over an additive category $\mathcal{C}$. The relation $\sim$ on the set of* $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X, Y)$ *of chain maps from X to Y is an equivalence relation, compatible with sums and compositions of chain maps.*

Proof. Denote by δ, ϵ the differentials of X, Y. Let f, f', $f'' : X \to Y$ be chain maps. The relation $\sim$ is reflexive: we have $f \sim f$, using the zero homotopy. The relation $\sim$ is symmetric: if $f \sim f'$ then $f' \sim f$; indeed, if h is a homotopy satisfying $f - f' = h \circ \delta + \epsilon \circ h$, then $f' - f =$

$(-h)\circ\delta+\epsilon\circ(-h)$. Finally, the relation $\sim$ is transitive: if $f-f'=h\circ\delta+\epsilon\circ h$ for some homotopy h and $f'-f''=k\circ\delta+\epsilon\circ k$ for some homotopy k, then $h+k$ is a homotopy from X to Y satisfying $f-f''=f-f'+f'-f''=(h+k)\circ\delta+\epsilon\circ(h+k)$. If g, $g':X\to Y$ are chain maps and if $f\sim f'$ and $g\sim g'$, then $f+g\sim f'+g'$; this follows from taking sums of homotopies. If (Z,ζ) is a third chain complex and $g,g:Y\to Z$ are chain maps such that $f\sim f'$ and $g\sim g'$ via homotopies $h:X\to Y$ and $h':Y\to Z$, then a short calculation shows that $g\circ f\sim g'\circ f'$ via the homotopy $g\circ h+h'\circ f'$. □

Proposition 1.3.5. *Let A be an algebra over a commutative ring k and let f, $f':(X,\delta)\to(Y,\epsilon)$ be chain maps of complexes of A-modules.*

(i) For any homotopy $h:X\to Y$, the graded morphism $h\circ\delta+\epsilon\circ h:X\to Y$ is a chain map inducing the zero morphism from $H_*(X)$ to $H_*(Y)$.

(ii) If $f\sim f'$ then $H(f)=H(f'):H_*(X)\to H_*(Y)$.

(iii) If f is a homotopy equivalence, then f is a quasi-isomorphism.

(iv) If $X\simeq 0$ then X is acyclic.

Proof. Using $\delta\circ\delta=0$ and $\epsilon\circ\epsilon=0$, we have

$$\epsilon\circ(h\circ\delta+\epsilon\circ h)=\epsilon\circ h\circ\delta=(h\circ\delta+\epsilon\circ h)\circ\delta\ ,$$

hence $h\circ\delta+\epsilon\circ h$ is a chain map from X to Y. Moreover, the induced map

$$\ker(\delta)\to\ker(\epsilon)$$

by $h\circ\delta+\epsilon\circ h$ is equal to the map induced by $\epsilon\circ h$ and hence has image contained in $\mathrm{Im}(\epsilon)\subset\ker(\epsilon)$, which shows that it induces the zero map on homology. This proves (i). If $f\sim f'$ then by (i), the difference $f-f'$ induces the zero map on homology and thus $H(f)=H(f')$, which proves (ii). Suppose f has a homotopy inverse g. Then, by (ii), we have $\mathrm{Id}_{H_*(X)}=H(g\circ f)=H(g)\circ H(f)$, thus $H(g)$ and $H(f)$ are inverse, proving (iii). If $X\simeq 0$, then X is quasi-isomorphic to zero by (iii), which is equivalent to $H_*(X)=0$, whence (iv). □

Remark 1.3.6. Let A, B be two algebras over a commutative ring k and let $\mathcal{F}:\mathrm{Mod}(A)\to\mathrm{Mod}(B)$ a k-linear (not necessarily exact) functor. Since $\mathcal{F}$ sends A-modules to B-modules and A-homomorphisms to B-homomorphisms, it extends to a functor, denoted by the same letter, $\mathcal{F}:\mathrm{Ch}(\mathrm{Mod}(A))\to\mathrm{Ch}(\mathrm{Mod}(B))$. This functor need not send a quasi-isomorphism to a quasi-isomorphism. But it sends a homotopy $h:X\to Y$ between chain complexes of A-modules X and Y to a homotopy $\mathcal{F}(h):\mathcal{F}(X)\to\mathcal{F}(Y)$ between the chain complexes of B-modules $\mathcal{F}(X)$ and $\mathcal{F}(Y)$. Thus homotopic chain maps from $X\to Y$ are sent to homotopic chain maps from $\mathcal{F}(X)\to\mathcal{F}(Y)$. In particular, $\mathcal{F}$ sends a homotopy equivalence $X\simeq Y$ to a homotopy equivalence $\mathcal{F}(X)\simeq\mathcal{F}(Y)$, and hence $\mathcal{F}$ sends contractible chain complexes of A-modules to contractible chain complexes of B-modules.

Definition 1.3.7. Let A be an algebra over a commutative ring k. The *homotopy category of complexes over* $\mathrm{Mod}(A)$ is the category $K(\mathrm{Mod}(A))$ whose objects are the complexes over $\mathrm{Mod}(A)$ and and whose morphisms are the homotopy equivalence classes

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(X,Y)=\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X,Y)/\sim$$

of chain maps, for any two complexes X, Y over $\mathcal{C}$. The composition of morphisms in $K(\mathrm{Mod}(A))$ is induced by that in $\mathrm{Ch}(\mathrm{Mod}(A))$. We denote by $K^+(\mathrm{Mod}(A))$, $K^-(\mathrm{Mod}(A))$, $K^b(\mathrm{Mod}(A))$ the full subcategories of $K(\mathrm{Mod}(A))$ consisting of left bounded, right bounded, bounded complexes of A-modules, respectively.

More explicitly, the composition in $K(\mathrm{Mod}(A))$ is defined as follows. If $f : X \to Y$ and $g : Y \to Z$ are chain maps, and if we denote by $[f]$ the class of all chain maps homotopic to f, then $[f] : X \to Y$ is a morphism in the category $K(\mathrm{Mod}(A))$. Similarly for $[g]$. We define the composition of $[f]$ and $[g]$ in $K(\mathrm{Mod}(A))$ by $[g] \circ [f] = [g \circ f]$. For this to be well-defined we need the observation from Proposition 1.3.4 that $\sim$ is compatible with the composition of chain maps. If f, $f' : X \to Y$ are chain maps, then the equality $[f] = [f']$ is equivalent to $f - f' \sim 0$. Thus if we denote by $\mathrm{Hom}^0_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)$ the k-submodule of all chain maps $f : X \to Y$ satisfying $f \sim 0$, or equivalently, $[f] = [0]$, then $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, Y)$ is the quotient space

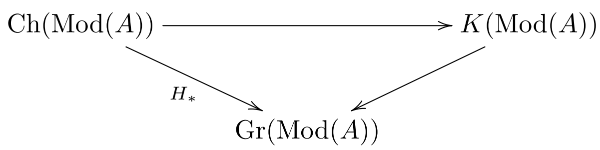
$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, Y) = \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)/\mathrm{Hom}^0_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)\ .$$

Note that if $f \sim 0$ or $g \sim 0$, then $g \circ f \sim 0$. Applied to $f = \mathrm{Id}_X$ or $g = \mathrm{Id}_Y$ this implies that if one of X, Y is contractible, then $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, Y) = 0$.

The categories $K(\mathrm{Mod}(A))$ and $\mathrm{Ch}(\mathrm{Mod}(A))$ have the same objects, but morphisms in $K(\mathrm{Mod}(A))$ are classes of morphisms in $\mathrm{Ch}(\mathrm{Mod}(A))$. Thus we have a functor

$$\mathrm{Ch}(\mathrm{Mod}(A)) \to K(\mathrm{Mod}(A))$$

which sends any chain complex X to itself and any chain map $f : X \to Y$ to the homotopy class $[f]$ of f. Taking homology yields a functor $H_* : \mathrm{Ch}(\mathrm{Mod}(A)) \to \mathrm{Gr}(\mathrm{Mod}(A))$. It follows from Proposition 1.3.5 that for f a chain map, the induced map $H(f)$ on homology depends only on the homotopy class $[f]$ of f, and therefore this functor factors through the canonical functor $\mathrm{Ch}(\mathrm{Mod}(A)) \to K(\mathrm{Mod}(A))$; that is, we have a commutative diagram of canonical functors



If two chain maps f, $f' : X \to Y$ of complexes of A-modules are homotopy equivalent via a homotopy $h : X \to Y$, then for any integer i, the "shifted" chain maps $f[i], f'[i] : X[i] \to Y[i]$ are homotopic via the homotopy $h[i]$ given by $h[i]_n = h_{n-i}$ for any $n \in \mathbb{Z}$. In other words, the shift automorphism $[i]$ of $\mathrm{Ch}(\mathrm{Mod}(A))$ induces an automorphism, still denoted by $[i]$, of the homotopy category $K(\mathrm{Mod}(A)))$. This automorphism preserves any of the subcategories $K^+(\mathrm{Mod}(A))$, $K^-(\mathrm{Mod}(A))$, $K^b(\mathrm{Mod}(A))$.

The following theorem shows that although a bounded below complex of projective A-modules is not a projective object in the category of chain complexes, it does have a lifting property with respect to quasi-isomorphisms. Similarly, bounded above complexes of injective A-modules have the extension property with respect to quasi-isomorphisms. See the Definition 6.3.1 for a definition of projective and injective objects in an arbitrary category, and the Theorems 6.3.6 and 6.3.8 for characterisations of projective and injective objects in module categories.

Theorem 1.3.8 (Lifting/extending chain maps). *Let A be an algebra over a commutative ring k. Let P be a complex of projective A-modules, I a complex of injective objects A-modules, and let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of complexes of A-modules.
(i) Suppose that X is acyclic and that one of P, Y is bounded below. The map

$$\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P, Y) \to \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P, Z)$$

given by composition with g is surjective and induces an isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Y) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Z)\ .$$

(ii) Suppose that Z is acyclic and that one of Y, I is bounded above. The map

$$\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(Y, I) \to \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, I)$$

given by precomposition with f is surjective and induces an isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(Y, I) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, I)\ .$$

Proof. (i) Denote by δ, ϵ, ζ, π the differentials of X, Y, Z, P, respectively. We first show the surjectivity of the map

$$\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P, Y) \to \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P, Z)$$

sending a chain map $p : P \to Y$ to the chain map $g \circ p : X \to Z$. Let $q : P \to Z$ be a chain map. We need to construct a chain map $p : P \to Y$ satisfying $g \circ p = q$. We construct such a chain map p inductively, by induction over the degree. In order to start an inductive argument, we will need the hypothesis that one of P or Y is bounded below. (Note that if Y is bounded below, then so is Z by the surjectivity of g.) This hypothesis ensures that we have $q_i = 0$ for all sufficiently small integers i, so we may simply take $p_i = 0$ for i sufficiently small. Let n be an integer. Suppose we have already constructed morphisms $p_i : P_i \to Y_i$ satisfying

$$g_i \circ p_i = q_i$$

$$\epsilon_i \circ p_i = p_{i-1} \circ \pi_i$$

for $i < n$. We construct p_n as follows. Since g_n is an epimorphism and P_n is projective, there is a morphism $p'_n : P_n \to Y_n$ such that $g_n \circ p'_n = q_n$. That is, p'_n satisfies the first of the two conditions above, but we may have to adjust p'_n to make sure, that it is compatible with the differentials as in the second condition. We have

$$g_{n-1} \circ (\epsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

because q is a chain map. Thus we have

$$\mathrm{Im}(\epsilon_n \circ p'_n - p_{n-1} \circ \pi_n) \subset \ker(g_{n-1}) = \mathrm{Im}(f_{n-1})$$

Consequently, since f_{n-1} is a monomorphism, there is a morphism $\sigma : P_n \to X_{n-1}$ such that

$$f_{n-1} \circ \sigma = \epsilon_n \circ p'_n - p_{n-1} \circ \pi_n$$

Moreover, we have

$$f_{n-2} \circ \delta_{n-1} \circ \sigma = \epsilon_{n-1} \circ f_{n-1} \circ \sigma = \epsilon_{n-1} \circ \epsilon_n \circ p'_n - \epsilon_{n-1} \circ p_{n-1} \circ \pi_n = -p_{n-2} \circ \pi_{n-1} \circ \pi_n = 0$$

and hence

$$\delta_{n-1} \circ \sigma = 0$$

as f_{n-2} is a monomorphism. Therefore we have

$$\mathrm{Im}(\sigma) \subset \ker(\delta_{n-1}) = \mathrm{Im}(\delta_n) \ ,$$

where the last equality holds as X is acylic. Since P_n is projective, the morphism $\sigma : P_n \to \mathrm{Im}(\delta_n)$ lifts to a morphism $\rho : P_n \to X_n$; that is, $\sigma = \delta_n \circ \rho$. Set $p_n = p'_n - f_n \circ \rho$. We still have

$$g_n \circ p_n = g_n \circ p'_n - g_n \circ f_n \circ \rho = g_n \circ p'_n = q_n \ ,$$

and we now also have the compatibility with the differentials

$$\epsilon_n \circ p_n = \epsilon_n \circ p'_n - \epsilon_n \circ f_n \rho = \epsilon_n \circ p'_n - f_{n-1} \circ \delta_n \circ \rho = \epsilon_n \circ p'_n - f_{n-1} \circ \sigma = \epsilon_n \circ p'_n - (\epsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = p_{n-1} \circ \pi_n$$

as required. This shows the surjectivity of the map given by composition with g.

We need to show that $p \sim 0$ if and only if $q \sim 0$. If $p \sim 0$ there is a homotopy $h : P \to Y$ such that

$$p = \epsilon \circ h + h \circ \pi \ .$$

Composing with g yields

$$q = g \circ p = g \circ \epsilon \circ h + g \circ h \circ \pi = \delta \circ g \circ h + g \circ h \circ \pi \ ,$$

where we have used that g is a chain map. Thus $q \sim 0$ via the homotopy $g \circ h : P \to X$. Conversely, suppose that $q \sim 0$. We need to show that then $p \sim 0$. The first part of the argument shows that we may assume that $q = 0$. To see this, observe first that since g_{n+1} is an epimorphism, any morphism $P_n \to X_{n+1}$ lifts to a morphism $P_n \to Y_{n+1}$, and thus every homotopy $P \to X$ lifts to some homotopy $P \to Y$. This means that if $q \sim 0$, there is some chain map $p' : P \to Y$ such that $p' \sim 0$ and $g \circ p' = q$, but p' need not be equal to p. It suffices to show that $p - p' \sim 0$. Since $g \circ (p - p') = 0$, we may therefore assume that $q = 0$. Then $g \circ p = q = 0$, hence

$$\mathrm{Im}(p) \subset \ker(g) = \mathrm{Im}(f) \ .$$

This implies that there is a chain map $u : P \to X$ such that $f \circ u = p$. It suffices to show that $u \sim 0$. This is again done inductively. Given an integer n, suppose that we have morphisms $h_i : P_i \to X_{i+1}$ satisfying $u_i = \delta_{i+1} \circ h_i + h_{i-1} \circ \pi_i$ for any $i < n$. Using this equality for $i = n - 1$ we get

$$\delta_n \circ (u_n - h_{n-1} \circ \pi_n) = \delta_n \circ u_n - \delta_n \circ h_{n-1} \circ \pi_n = \delta_n \circ u_n - (u_{n-1} - h_{n-2} \circ \pi_{n-1}) \circ \pi_n = \delta_n \circ u_n - u_{n-1} \circ \pi_n = 0 \ .$$

as u is a chain map. Thus

$$\mathrm{Im}(u_n - h_{n-1} \circ \pi_n) \subset \ker(\delta_n) = \mathrm{Im}(\delta_{n+1}) \ .$$

As P_n is projective, there is $h_n : P_n \to X_{n+1}$ such that

$$\delta_{n+1} \circ h_n = u_n - h_{n-1} \circ \pi_n$$

as required. This completes the proof of (i). The proof of (ii) is obtained by dualising the arguments (reversing all arrows and exchanging monomorphisms and epimorphisms). □

The above theorem holds verbatim for arbitrary abelian categories instead of module categories; the only adjustment in the proof is that inclusion maps need to replaced by canonical monomorphisms.

Remark 1.3.9. The condition that X is acyclic is equivalent to g being a quasi-ismorphism, by Corollary 1.2.9. In the statement of Theorem 1.3.8, the chain map g is in addition surjective in each degree. One can show that the second isomorphism in (i) holds for an arbitrary quasi-isomorphism g; that is, the surjectivity of g is not necessary for this isomorphism to hold. Similarly, the second isomorphism in (ii) holds for any quasi-isomorphism f; that is, the injectivity of f is not needed. We will see later that quasi-isomorphisms are in fact characterised by these isomorphisms for all bounded below (resp. bounded above) complexes of projective (resp. injective) A-modules. See Theorem 1.3.16 below. We will come back to this in Section 4.2, where we show that homotopy categories are triangulated.

Corollary 1.3.10. *Let A be an algebra over a commutative ring k.*

(i) A chain complex X of A-modules is acyclic if and only if $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, X) = \{0\}$ for any bounded below chain complex P of projective A-modules.

(ii) A cochain complex Y is acyclic if and only if $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(Y, I) = \{0\}$ for any bounded below cochain complex I of injective A-modules.

Proof. Let X be an acyclic chain complex of A-modules. Applying Theorem 1.3.8 to the short exact sequence of complexes

$$0 \longrightarrow X \xrightarrow{\mathrm{Id}_X} X \longrightarrow 0 \longrightarrow 0$$

shows that then $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, X) = \{0\}$ for any bounded below complex P of projective A-modules. If X is a chain complex of A-modules which is not acyclic, then there is an integer n such that $H_n(X)$ is not zero, or equivalently, such that the canonical monomorphism $\mathrm{Im}(\delta_{n+1}) \subset \ker(\delta_n)$ is not an isomorphism, where δ is the differential of X. Let P be the complex which is zero in any degree other than n and which is a projective A-module such that there is an epimorphism $\pi : P_n \to \ker(\delta_n)$. Then π defines a chain map from P to X which cannot be homotopic to zero, because π does not factor through δ_{n+1}. This shows (i). By dualising the above proof, one shows (ii). □

This proof, suitably adapted, shows more generally that for any acyclic chain complex X and any acyclic cochain complex Y over an abelian category $\mathcal{C}$, the spaces $\mathrm{Hom}_{K(\mathcal{C})}(P, X)$ and

$\mathrm{Hom}_{K(\mathcal{C})}(X, I)$ are zero, for any bounded below complex P of projective objects in $\mathcal{C}$ and any bounded below cochain complex I of injective objects in $\mathcal{C}$. The converse, as stated in Corollary 1.3.10, holds if $\mathcal{C}$ has enough projective and injective objects.

The following two observations describe the homology and cohomology of complexes in terms of homotopy classes of chain maps. Both of these observations are very easy - but they have an important consequence: since chain maps can be composed, the interpretation of (co-)homology in terms of chain maps introduces extra structure on (co-)homology. We will see later that the graded algebra structure of Ext-algebras is induced in this way.

Proposition 1.3.11. *Let X be a chain complex of A-modules and let n be an integer. There is a natural isomorphism*

$$H_n(X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A[n], X) \ ,$$

where $A[n]$ is the complex equal to A in degree n and zero in all other degrees.

Proof. A chain map $A[n] \to X$ is represented by a commutative diagram of the form

$$\begin{CD} \cdots @>>> 0 @>>> A @>>> 0 @>>> \cdots \\ @. @VVV @VV{f}V @VVV @. \\ \cdots @>>> X_{n+1} @>>{\delta_{n+1}}> X_n @>>{\delta_n}> X_{n-1} @>>> \cdots \end{CD}$$

for some A-homomorphism $f : A \to X_n$ satisfying $\delta_n \circ f = 0$, where δ is the differential of X. Thus any such A-homomorphism f maps A to $\ker(\delta_n)$, and and any such A-homomorphism is uniquely determined by the the image of 1_A in $\ker(\delta_n)$. A homotopy from $A[n]$ to X is zero except in degree n, where it is a map $A \to X_{n+1}$. The chain map determined by the homomorphism f is homotopic to zero if and only if f factors through δ_{n+1}. A necessary condition for that to happen is that $\mathrm{Im}(f) \subseteq \mathrm{Im}(\delta_{n+1})$. This condition is also sufficient because A is projective as an A-module, so every A-homomorphism $A \to \mathrm{Im}(\delta_{n+1})$ lifts through the surjective map $X_{n+1} \to \mathrm{Im}(\delta_{n+1})$. Thus the map sending $x \in \ker(\delta_n)$ to the unique chain map $A[n] \to X$ sending 1_A to x induces an isomorphism as stated. The naturality statements just means that this defines an isomorphism of functors $H_n(-) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A[n], -)$ from $\mathrm{Ch}(\mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$; this is an easy verification. The result follows. □

For (X, δ) a chain complex of A-modules and V an A-module, applying the contravariant functor $\mathrm{Hom}_A(-, V)$ to (X, δ), written as a chain of morphisms,

$$\cdots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \longrightarrow \cdots$$

yields a cochain complex $\mathrm{Hom}_A(X, V)$ of k-modules of the form

$$\cdots \longrightarrow \mathrm{Hom}_A(X_{n-1}, V) \xrightarrow{\delta^{n-1}} \mathrm{Hom}_A(X_n, V) \xrightarrow{\delta^n} \mathrm{Hom}_A(X_{n+1}, V) \longrightarrow \cdots \ ;$$

that is, the cochain complex $\mathrm{Hom}_A(X, V)$ is defined by

$$\mathrm{Hom}_A(X, V)^n = \mathrm{Hom}_A(X_n, V)$$

with differential

$$\delta^n : \mathrm{Hom}_A(X_n, V) \to \mathrm{Hom}_A(X_{n+1}, V)$$

given by $\delta^n(\alpha) = \alpha \circ \delta_{n+1}$ for any $\alpha \in \mathrm{Hom}_A(X_n, V)$. The cohomology of this cochain complex is as follows.

Proposition 1.3.12. *Let X be a chain complex of A-modules, and let V be an A-module. For any integer n we have a natural isomorphism of k-modules*

$$H^n(\mathrm{Hom}_A(X, V)) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, V[n]) \ .$$

Proof. A chain map from X to $V[n]$ is a commutative diagram of the form

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_{n+1} & \xrightarrow{\delta_{n+1}} & X_n & \xrightarrow{\delta_n} & X_{n-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & V & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

This is a chain map if and only if $\alpha \circ \delta_{n+1} = 0$, thus if and only of $\delta^n(\alpha) = 0$. This shows that

$$\ker(\delta^n) = \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, V[n]) \ .$$

Any homotopy between the above complexes is zero except possibly in degree $n-1$, where it is homomorphism $h : X_{n-1} \to V$. Thus α determines a chain map which is homotopic to zero if and only if $\alpha = h \circ \delta_n$ for some $h : X_{n-1} \to V$; that is, if and only if $\alpha \in \mathrm{Im}(\delta^{n-1})$ The naturality statements is easily verified; this means that the contravariant functors $H^n(\mathrm{Hom}_A(-, V))$ and $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(-, V[n])$ from $\mathrm{Ch}(\mathrm{Mod}(A))$ to $\mathrm{Mod}(k)$ are isomorphic. □

The right side in the isomorphism in this proposition depends only on the homotopy category $K(\mathrm{Mod}(A))$, so any isomorphism in this category preserves the left side as well:

Corollary 1.3.13. *Let $f : X \to Y$ be a chain homotopy equivalence of chain complexes of A-modules, let n be an integer and V an A-module. Then f induces an isomorphism*

$$H^n(\mathrm{Hom}_A(Y, V)) \cong H^n(\mathrm{Hom}_A(X, V)) \ .$$

Definition 1.3.14. Let (X, δ) be a chain complex of A-modules. The *cone of X*, is the chain complex $C(X)$ defined as follows. For any integer n we set

$$C(X)_n = X_{n-1} \oplus X_n .$$

The differential of $C(X)$ in degree n is given by

$$\Delta_n = \begin{pmatrix} -\delta_{n-1} & 0 \\ \mathrm{Id}_{X_{n-1}} & \delta_n \end{pmatrix} : X_{n-1} \oplus X_n \to X_{n-2} \oplus X_{n-1} \ ;$$

that is, $\delta_n(x, y) = (-\delta_{n-1}(x), x + \delta_n(y))$ for any $x \in X_{n-1}$ and any $y \in X_n$.

The sign of $-\delta_{n-1}$ in the definition of Δ_n ensures that Δ is indeed a differential; that is, $\Delta_{n-1} \circ \Delta_n = 0$.

Proposition 1.3.15. *Let X be a chain complex of A-modules. The cone $C(X)$ is a contractible chain complex. The canonical inclusions $X_n \to X_{n-1} \oplus X_n$ define a chain map $i_X : X \to C(X)$, and the canonical projections $X_{n-1} \oplus X_n \to X_{n-1}$ define a chain map $C(X) \to X[1]$ such that the sequence of chain maps*

$$0 \longrightarrow X \xrightarrow{i_X} C(X) \xrightarrow{p_X} X[1] \longrightarrow 0$$

is exact. In particular, X is isomorphic to a subcomplex of a contractible complex as well as a quotient of a contractible complex.

Proof. Consider a homotopy which identifies the summand X_n of $C(X)_n$ with the summand X_n of $C(X)_{n+1}$. One checks that this homotopy implies that $C(X)$ is contractible. The rest is obvious. □

Note that for p_X to be a chain map, we need the earlier sign convention by which the differential of the complex $X[1]$ is the negative of the shifted differential of X.

Theorem 1.3.16. *Let $f : X \to Y$ be a chain map of complexes of A-modules. The following are equivalent.*

(i) f is a quasi-isomorphism.

(ii) For every bounded below chain complex of projective A-modules P, composition with f induces an isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Y) \ .$$

(iii) For every bounded above chain complex of injective A-modules I, precomposition with f induces an isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(Y, I) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, I) \ .$$

Proof. We may add any contractible complex to X without changing the statement. Since $C(Y)$ is contractible, by adding $C(Y)$ to X, we may assume that f is surjective. By Theorem 1.3.8, f is a quasi-isomorphism if and only if $K = \ker(f)$ is acyclic. By Corollary 1.3.10 this is the case if and only if $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, X) = \{0\}$ for any bounded below chain complex P of projective A-modules. Thus Theorem 1.3.8 shows that (i) implies (ii). For the converse, if f is not a quasi-isomorphism, then there is an integer n such that $H_n(f) : H_n(X) \to H_n(Y)$ is not an isomorphism. It follows from Proposition 1.3.11 that the map

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(A[n], X) \to \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A[n], Y)$$

inducd by f is not an isomorphism, where $A[n]$ is the complex equal to A in degree n and zero in all other degrees. Since $A[n]$ is a bounded complex of projective A-modules, this shows that (ii) implies (i). Arguing similarly adding the cone $C(X)$ to Y, we may assume that f is injective. Using $\mathrm{coker}(f)$ instead of $\ker(f)$ and the fact that the module category of A has enough injectives, a variation of the previous arguments shows the equivalence of (i) and (iii). □

Exercise 1.3.17. Show that a chain complex over some additive category of the form

$$\cdots \longrightarrow 0 \longrightarrow U \xrightarrow{\mathrm{Id}_U} U \longrightarrow 0 \longrightarrow \cdots$$

is contractible, where the two terms equal to the same object U are in two arbitrary consecutive degrees. (One can show that any contractible chain complex is a direct sum of complexes of this form.)

Exercise 1.3.18. Show that a short exact sequence of A-modules is split if and only if it is contractible when regarded as a chain complex.

Exercise 1.3.19. Show that a direct summand (in the category of chain complexes over some additive category) of a contractibe complex is contractible.

Exercise 1.3.20. Show that he forgetful functor $\mathrm{Ch}(\mathrm{Mod}(A)) \to \mathrm{Gr}(\mathrm{Mod}(A))$ sends the exact sequence of chain complexes

$$0 \longrightarrow X \xrightarrow{i_X} C(X) \xrightarrow{p_X} X[1] \longrightarrow 0$$

from Proposition 1.3.15 to a split exact sequence of graded A-modules. Give an example where this exact sequence does not split as a sequence of chain complexes. Show that this exact sequence is split as a sequence of chain complexes, if and only if X is contractible.

Exercise 1.3.21. Use the previous exercise to show that a chain complex of A-modules is projective as an object of the category of chain complexes $\mathrm{Ch}(\mathrm{Mod}(A))$ if and only if it is a contractible complex of projective A-modules.

Remark 1.3.22. Let A be an algebra over a commutative ring. Two homotopic chain maps f, $f' : X \to Y$ between complexes of A-modules X, Y may have different kernels and cokernels. Therefore, in the category $K(\mathrm{Mod}(A))$ there is no well-defined notion of kernel and cokernel of a morphism. In particular, the category $K(\mathrm{Mod}(A))$ is additive but not abelian - there is no notion of exactness. The search for a replacement of short exact sequences is what led to the concept of a *triangulated category.*

Chapter 2

Ext and Tor, derived categories and functors

2.1 Ext and Tor

Let A be an algebra over a commutative ring k. Informally, a bounded below chain complex of A-modules of the form

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\pi} U \longrightarrow 0$$

is called a *projective resolution of* U if it is exact and all P_i are projective. The above exact complex can be viewed as a chain map obtained from 'bending down' the map π and viewing U as a chain complex concentrated in degree zero:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{\delta_2} & P_1 & \xrightarrow{\delta_1} & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & U & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The exactness of the above sequence is equivalent to this chain map being a quasi-isomorphism because the homology of both rows is concentrated in degree 0, where it is isomorphic to $P_0/\mathrm{Im}(\delta_0) = P_0/\ker(\pi) \cong U$. If no confusion ariese, we will denote the A-homomorphism $P_0 \to U$ and the induced chain map $P \to U$ by the same letter.

Similarly, the informal version of an *injective resolution of* U is an exact bounded below cochain complex of the form

$$0 \longrightarrow U \xrightarrow{\iota} I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} I^2 \xrightarrow{\delta^2} \cdots$$

where the modules I^i are injective. As before, we view the A-homomorphism ι as a cochain map, and the exactness of the previous cochain complex implies that ι yields a quasi-isomorphism of cochain complexes, again denoted by the same letter whenever convenient,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & U & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \xrightarrow{\delta^0} & I^1 & \xrightarrow{\delta^1} & \cdots \end{array}$$

The formal definitions of projective and injective resolutions are as follows.

Definition 2.1.1. Let A be an algebra over a commutative ring k. A *projective resolution of an A-module* U is a pair (P, μ) consisting of a complex P of projective A-modules such that $P_i = 0$ for $i < 0$ and a quasi-isomorphism $\mu : P \to U$.

If μ is clear from the context or not needed in a particular statement, we suppress it and simply say 'Let P be a projective resolution of U...', implicitly assuming that there is such a quasi-isomorphism μ.

Definition 2.1.2. Let A be an algebra over a commutative ring k. An *injective resolution of an A-module* U is a pair (I, ι) consisting of a cochain complex I of injective A-modules such that $I^i = 0$ for $i < 0$ and a quasi-isomorphism $\iota : U \to I$.

Every A-module has a projective resolution P; in fact, the resolution can be taken to be *free*; that is, all P_i are free A-modules. This follows from the fact that every A-module is a quotient of a free A-module. Thus there is a free A-module P_0 such there exists a surjective A-homomorphism $\mu : P_0 \to U$. Applied to $\ker(\mu)$, there exists a free A-module P_1 and a surjective A-homomorphism $\delta_1 : P_1 \to \ker(\mu)$. One constructs P inductively by taking for P_n a free A-module and for δ_n a surjective A-homomorphism $P_n \to \ker(\delta_{n-1})$ composed with the inclusion $\ker(\delta_{n-1}) \subseteq P_{n-1}$, where $n \geq 2$. Every A-module also has an injective resolution. This follows from the fact that every A-module is a submodule of an injective A-module, and hence dualising the construction of a projective resolution one can construct an injective resolution inductively.

Examples 2.1.3.

(1) Let n be a positive integer. We have an obvious exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{a \mapsto an} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

Thus a projective resolution of $\mathbb{Z}/n\mathbb{Z}$ is the pair consisting of the 2-term complex $\mathbb{Z} \xrightarrow{a \mapsto an} \mathbb{Z}$ with nonzero differential given by multiplication with n, together with the canonical map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ starting at the second term of this complex.

(2) An abelian group A is called *divisible* if for any $a \in A$ and any positive integer n there is $b \in A$ such that $nb = a$. One can show that the divisible abelian groups are exactly the injective $\mathbb{Z}$-modules. In particular, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules. The obvious short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

shows that an injective resolution of $\mathbb{Z}$ is the pair consisting of the 2-term complex $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, together with the inclusion $\mathbb{Z} \to \mathbb{Q}$. For n a positive integer, the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by

$\frac{1}{n}+\mathbb{Z}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. The group $\mathbb{Q}/\mathbb{Z}$ is the colimit of these subgroups, and every element in $\mathbb{Q}/\mathbb{Z}$ has finite order; that is, $\mathbb{Q}/\mathbb{Z}$ is a torsion abelian group. By contrast, the torsion subgroup of $\mathbb{Q}$ is trivial.

(3) Hilbert's Syzygy Theorem shows that every module U over a polynomial algebra $k[x_1, x_2, .., x_n]$ in $n \geq 1$ indeterminates over a field k has a projective resolution of length at most n (that is, with at most $n+1$ nonzero terms). For $n = 1$ this follows from tensoring the obvious short exact sequence

$$0 \longrightarrow k[x] \xrightarrow{f \mapsto xf} k[x] \longrightarrow k \longrightarrow 0$$

by $- \otimes_k U$. This yields a short exact sequence in which the first two terms are the free $k[x]$-modules $k[x] \otimes_k U$, and the third term is $k \otimes_k U \cong U$. For $n > 1$, one way to see this is to note that $k[x_1, x_2, .., x_n]$ is isomorphic to the tensor product of the algebras $k[x_i]$ and then tensor the above two-term complexes together for $1 \leq i \leq n$; this yields a complex with $n+1$ terms. (One needs to extend the tensor product to complexes for this argument).

(4) Here is an example of a projective resolution of infinite length. Let k be a commutative ring. Set $A = k[x]/(x^2)$. That is, A has a k-basis $\{1, \bar{x}\}$ such that $\bar{x}^2 = 0$. Note that $A\bar{x}$ is an ideal in A and that $A/A\bar{x} \cong k$, viewed as an A-module with $\bar{x}$ acting as zero on k. Multiplication by $\bar{x}$ on A is an endomorphism of A with image $A\bar{x}$ and also kernel $A\bar{x}$, since $\bar{x}^2 = 0$. Moreover, we have $A\bar{x} \cong k$. Thus we get an infinite projective resolution of the form

$$\cdots \longrightarrow A \xrightarrow{\bar{x}} A \xrightarrow{\bar{x}} k \longrightarrow 0$$

where the superscript $\bar{x}$ means multiplication by $\bar{x}$. One can in fact show that any projective resolution of k is infinite.

Definition 2.1.4. Let A be an algebra over a commutative ring k. Let U, V be A-modules. For any nonnegative integer n we define a k-module $\mathrm{Ext}^n_A(U, V)$ as follows. Let P be a projective resolution of U with differential π. Applying $\mathrm{Hom}_A(-, V)$ yields a cochain complex $\mathrm{Hom}_A(P, V)$

$$\mathrm{Hom}_A(P_0, V) \xrightarrow{\pi^0} \mathrm{Hom}_A(P_1, V) \xrightarrow{\pi^1} \mathrm{Hom}_A(P_2, V) \longrightarrow \cdots$$

that is, $\mathrm{Hom}_A(P, V)$ is in degree $n \geq 0$ equal to $\mathrm{Hom}_A(P_n, V)$ with differential $\pi^n : \mathrm{Hom}_A(P_n, V) \to \mathrm{Hom}_A(P_{n+1}; V)$ given by $\pi^n(\alpha) = \alpha \circ \pi_{n+1}$ for $n \geq 0$. We set

$$\mathrm{Ext}^n_A(U, V) = H^n(\mathrm{Hom}_A(P, V)) \ .$$

Proposition 2.1.5. *Let A be an algebra over a commutative ring k. Let U, V be A-modules, P a projective resolution of U, and let n be an integer. we have a natural isomorphism of k-modules*

$$\mathrm{Ext}^n_A(U, V) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, V[n]) \ .$$

Proof. This is a special case of Proposition 1.3.12. □

We will use Proposition 2.1.5 to show that $\mathrm{Ext}^n_A(U, V)$ does not depend on the choice of P and that $\mathrm{Ext}^n_A(U, V)$ is contravariant functorial in U and covariantly functorial in V.

Proposition 2.1.6. *Let A be an algebra over a commutative ring k. Let (P, μ), (Q, ν) be projective resolutions of A-modules U, V, respectively. We have canonical isomorphisms*

$$\mathrm{Hom}_A(U, V) \cong \mathrm{Ext}^0_A(U, V) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Q) \ .$$

This isomorphism sends $\alpha : U \to V$ to the homotopy class of a chain map $\varphi : P \to Q$ such that $\alpha \circ \mu \sim \nu \circ \varphi$ as chain maps from P to V.

Proof. With the notation of Definition 2.1.4, we have $\mathrm{Ext}^0_A(U, V) = \ker(\pi^0)$. This is the space of all A-homomorphisms $\alpha : P_0 \to V$ such that $\alpha \circ \pi_1 = 0$, that is, all A-homomorphisms α such that $\mathrm{Im}(\pi_1) \subseteq \ker(\alpha)$. Any such homomorphism factors uniquely through the canonical surjection $P_0 \to P_0/\mathrm{Im}(\pi_1)$. Denote by $\mu : P \to U$ a quasi-isomorphism; that is, μ is determined by a surjective A-homomorphism (still denoted μ) from P_0 to V such that $\ker(\mu) = \mathrm{Im}(\pi_1)$. Thus $\ker(\pi^0)$ can be canonically identified with the space of A-homomorphisms from $P/\ker(\mu) = U$ to V. This shows the first isomorphism. Since the projective resolution Q comes with a quasi-isomorphism $\nu : Q \to V$, composing with ν induces by Theorem 1.3.8 an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Q) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, V) = \mathrm{Ext}^0_A(U, V)$ The compatibility with μ and ν follows from the explicit descriptions of these two isomorphisms. □

There is a structural way to look at this result: attaching a projective resolution to an A-module is a functorial construction through which the module category $\mathrm{Mod}(A)$ gets fully embedded into the chain homotopy category $K^-(\mathrm{Proj}(A))$ of bounded below chain complexes in the category $\mathrm{Proj}(A)$ of projective A-modules.

We have an analogous result for injective resolutions.

Proposition 2.1.7. *Let A be an algebra over a commutative ring k. Let (I, ι), (J, κ) be injective resolutions of A-modules U, V, respectively. We have canonical isomorphisms*

$$\mathrm{Hom}_A(U, V) \cong \mathrm{Ext}^0_A(U, V) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(I, J) \ .$$

This isomorphism sends $\alpha : U \to V$ to the homotopy class of a chain map $\psi : I \to J$ such that $\kappa \circ \alpha \sim \psi \circ \iota$ as chain maps from U to J.

Proof. Dualise the proof of Proposition 2.1.6. □

The functoriality of sending a module to one of its projective resp. injective resolutions implies in particular that projective and injective resolutions are unique up to unique homotopy.

Proposition 2.1.8. *Let A be an algebra over a commutative ring k. For any two projective resolutions (P, μ), (P', μ') of an A-module U there is a homotopy equivalence $\beta : P \simeq P'$ such that $\mu' \circ \beta = \mu$. Moreover, β is unique up to homotopy.*

Proof. Applying Proposition 2.1.6 with $U = V$ and $P' = Q$ shows that Id_U corresponds to the homotopy class of a chain map $\beta : P \to P'$ satisfying $\mu' \circ \beta \sim \mathrm{Id}_U \circ \mu = \mu$. Exchanging the roles of P and P' yields a chain map, unique up to homotopy, $\gamma : P' \to P$ satisfying $\mu \circ \gamma = \mu'$. Thus $\mu \circ \gamma \circ \beta = \mu$. But also $\mu \circ \mathrm{Id}_P = \mu$. Since, again by Theorem 1.3.8, composition with μ induces an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, P) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, U)$, it follows that $\gamma \circ \beta \sim \mathrm{Id}_P$. A similar argument shows that $\beta \circ \gamma \sim \mathrm{Id}_{P'}$, and hence that $P \simeq P'$ as stated. □

We emphasise that this Corollary is more precise than merely stating that two projective resolutions P, P' of U are homotopy equivalent as chain complexes; it says that there is a homotopy equivalence $P \simeq P'$ which is unique up to homotopy with the property that it is compatible with the second component of what makes a projective resolution, namely the quasi-isomorphisms μ and μ'. That is, in an appropriate category of pairs consisting of a chain complex and a chain map from this complex to U, the pairs (P, μ) and (P', μ') are isomorphic up to unique isomorphism. We have similar results for injective resolutions.

Proposition 2.1.9. *Let A be an algebra over a commutative ring k. For any two injective resolutions (I, ι), (I', ι') of an A-module U there is a homotopy equivalence $\gamma : I \simeq I'$ such that $\gamma \circ \iota = \iota'$. Moreover, γ is unique up to homotopy.*

Proof. Dualise the proof of Proposition 2.1.8. □

Using the above results we interpret $\mathrm{Ext}_A^n(U, V)$ in terms of homotopy classes of chain maps.

Theorem 2.1.10. *Let A be an algebra over a commutative ring k. Let U, V be A-modules with projective resolutions P, Q, iand injective resolutions I, J, respectively. Let $n \geq 0$ be an integer. We have a natural k-linear isomorphism*

$$\begin{aligned}\mathrm{Ext}_A^n(U, V) &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, V[n]) \\ &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Q[n]) \\ &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, J[n]) \\ &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(U, J[n]) \\ &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(I, J[n])\end{aligned}$$

Proof. The first isomorphism is from Proposition 2.1.5. Let $\nu : Q \to V$ be a quasi-isomorphism, and ν is surjective. Then $\nu[n] : Q[n] \to V[n]$ is a quasi-isomorphism, hence induces by Theorem 1.3.8 an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Q[n]) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, V[n])$. This shows the second isomorphism. The third isomorphism is induced by the quasi-isomorphism $V[n] \to J[n]$, the fourth isomorphism is induced by the quasi-isomorphism $P \to U$, and the last isomorphism is induced by the quasi-isomorphism $U \to I$. The naturality is an easy verification. □

The isomorphisms in Theorem 2.1.10 are determined by the quasi-isomorphisms $P \to U \to I$ and $Q \to V \to J$, which are a structural part of the data which make up projective and injective resolutions. The interpretation of Ext in terms of homotopy classes of chain maps has one important consequence: we can compose chain maps, and this introduces ring and module structures on Ext-spaces as follows.

Proposition 2.1.11. *Let A be an algebra over a commutative ring k. Let U, V, W be A-modules. We have a graded k-linear product*

$$\mathrm{Ext}_A^*(U, V) \times \mathrm{Ext}_A^*(V, W) \longrightarrow \mathrm{Ext}_A^*(U, W)$$

defined as follows. Let P, Q, R be projective resolutions of U, V, W, respectively. For

$$\zeta \in \mathrm{Ext}_A^n(U, V) = \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Q[n])$$

$$\eta \in \mathrm{Ext}_A^m(V, W) = \mathrm{Hom}_{K(\mathrm{Mod}(A))}(Q, R[m])$$

define

$$\eta \cup \zeta \in \mathrm{Ext}_A^{n+m}(U, W) = \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, R[n+m])$$

as the composition of morphisms

$$\eta \cup \zeta = (-1)^n \eta[n] \circ \zeta$$

With this product, the graded k-module

$$\mathrm{Ext}_A^*(U, U) = \oplus_{n \geq 0} \ \mathrm{Ext}_A^n(U, U)$$

is a graded unital associative k-algebra, and the graded k-module

$$\mathrm{Ext}_A^*(U, V) = \oplus_{n \geq 0} \ \mathrm{Ext}_A^n(U, V)$$

is an $\mathrm{Ext}_A^*(V, V)$*-*$\mathrm{Ext}_A^*(U, U)$*-bimodule.*

Proof. Composition of morphisms in any category is associative. One checks that the associativity is compatible with the sign convention. The result follows. □

The product in $\mathrm{Ext}_A^*(U, U)$ is called *cup product*.

Remark 2.1.12. Theorem 2.1.10 shows that in the definition of $\mathrm{Ext}_A^n(U, V)$ we could have used injective resolutions of V instead of projective resolutions of U and would have ended up with the same concept. In some circumstances, calculating an injective resolution may be easier than calculating a projective resolution. There are cases - such as in the category of sheaves - where every object has an injective resolution but not a projective resolution.

Proposition 2.1.13. *Let A be an algebra over a commutative ring k, and let U, V be A-modules. If U is projective or if V is injective, then* $\mathrm{Ext}_A^n(U, V) = \{0\}$ *for any positive integer n.*

Proof. If U is projective, then it is its own projective resolution, concentrated in degree 0. If $n > 0$, then there is no nonzero chain map $U \to V[n]$. Similarly, if V injective, then it is its own injective resolution, concentrated in degree 0, whence the result. □

The bifunctors Ext_A^n, for $n \in \mathbb{Z}$, are defined using the bifunctor $\mathrm{Hom}_A(-,-)$ applied to appropriate resolutions and taking cohomology. A similar construction, using the bifunctor $- \otimes_A -$, yields the bifunctors $\mathrm{Tor}_n^A(-,-)$.

Definition 2.1.14. Let A be an algebra over a commutative ring k. Let V be an A-module and W a right A-module. Let Q be a projective resolution of V. For $n \geq 0$ we set

$$\mathrm{Tor}_n^A(W, V) = H_n(W \otimes_A Q) \ .$$

That is, $\mathrm{Tor}_n^A(W, V)$ is the homology in degree n of the chain complex $W \otimes_A Q$ obtained from applying the covariant functor $W \otimes_A -$ to the projective resolution Q of V. Since any two projective resolutions of V are homotopy equivalent and since any functor maps homotopy equivalent complexes to homotopy equivalent complexes, it follows as in the case of Ext that $\mathrm{Tor}_n^A(W, V)$ does not depend on the choice of Q, up to unique isomorphism. One can show that $\mathrm{Tor}_n^A(W, V)$ is also isomorphic to the homology in degree n of the chain complex $R \otimes_A V$ obtained from applying the functor $- \otimes_A V$ to a projective resolution of the right A-module W. The use of the notation 'Tor' for this concept comes from the following fact.

Theorem 2.1.15. *Let A be an abelian group. Then $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A)$ is isomorphic to the torsion subgroup of A.*

The proof of this Theorem uses the fact that $\mathrm{Tor}_1^{\mathbb{Z}}(-, A)$ commutes with colimits, together with the fact that $\mathbb{Q}/\mathbb{Z}$ is the colimit of its finite subgroups $\mathbb{Z} \cdot \frac{1}{n} + \mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$, with $n \in \mathbb{N}$, and the following calculations.

Lemma 2.1.16. *Let A be an abelian group and n a positive integer. The following hold.*

$$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n|, A) \cong A/nA,$$

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong \{a \in A \mid na = 0\},$$

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) = 0, \ i \geq 2.$$

Proof. The exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$ yields the 2-term projective resolution $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ of $\mathbb{Z}/n\mathbb{Z}$. Thus $\mathrm{Tor}_*^{\mathbb{Z}}$ is the homology of the 2-term chain complex $B \xrightarrow{n} B$ obtained from tensoring the previous resolution with $- \otimes_{\mathbb{Z}} B$. The result follows. □

Similar arguments yield the following result.

Lemma 2.1.17. *Let A, B be abelian groups. Then $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is a torsion abelian group, and $\mathrm{Tor}_i^{\mathbb{Z}}(A, B) = 0$ for $i \geq 2$.*

Proof. See [12, Proposition 3.1.2]. □

One of the applications of the material in this Section concerns notions of dimensions defined in (co-)homological terms.

Definition 2.1.18. Let A be an algebra over a commutative ring k. Let U be an A-module. The *projective dimension of* U, denoted by $\mathrm{pdim}(U)$, is the smallest nonnegative integer n such that U has a projective resolution P satisfying $P_i = 0$ for $i > n$, with the convention $\mathrm{pdim}(U) = \infty$ if every projective resolution of U is unbounded. The *injective dimension of* U, denoted by $\mathrm{idim}(U)$, is the smallest nonnegative integer n such that U has an injective resolution I satisfying $I^i = 0$ for $i > n$, with the convention $\mathrm{idim}(U) = \infty$ if every injective resolution of U is unbounbded. The *global dimension of* A is equal to

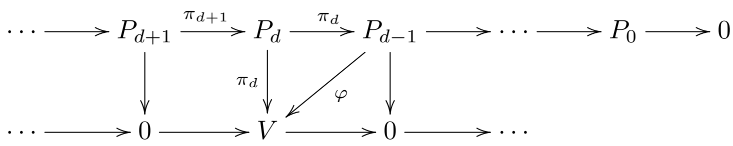
$$\mathrm{gldim}(A) = \sup\{\mathrm{pdim}(U) \mid U \in \mathrm{Mod}(A)\}$$

Thus $\mathrm{gldim}(A) = \infty$ unless every A-module U has a finite projective dimension whose length is bounded by some fixed integer, independent of U. Note that $\mathrm{gldim}(A)$ is defined in terms of left A-module; there is an obvious analogue of a right global dimension. Left and right global dimension of an algebra need not coincide. The following result shows that we could have defined the global dimension using injective dimensions of modules.

Theorem 2.1.19. *Let A be an algebra over a commutative ring k. We have*

$$\begin{aligned} \mathrm{gldim}(A) &= \sup\{\mathrm{idim}(U) \mid U \in \mathrm{Mod}(A)\} \\ &= \sup\{d \in \mathbb{N} \mid \mathrm{Ext}_A^d(U, V) \neq 0 \text{ for some } U, V \in \mathrm{Mod}(A)\} \end{aligned}$$

Proof. Let $d \geq 0$ and U, $V \in \mathrm{Mod}(A)$ such that $\mathrm{Ext}^d_A(U,V) \neq 0$. Let P be a projective resolution of U. Since $\mathrm{Ext}^d_A(U,V) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, V[d]) \neq 0$, there must be a nonzero A-homomorphism $P_d \to V$. In particular, $P_d \neq 0$. This shows $\mathrm{pdim}(U) \geq d$, hence $\mathrm{gldim}(A)$ is greater or equal to the supremum of all such d. Let I be an injective resolution of V. Since $\mathrm{Ext}^d_A(U,V) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(U, I[d]) \neq 0$, there must be a nonzero A-homomorphism $U \to I^d$. In particular, $I^d \neq 0$. This shows $\mathrm{idim}(U) \geq d$, hence $\sup\{\mathrm{idim}(U) \mid U \in \mathrm{Mod}(A)\}$ is greater or equal than the supremum of all such d. For the converse inequality, suppose that $d \geq 0$ satisfies $\mathrm{Ext}^d_A(U,V) = 0$ for all U, $V \in \mathrm{Mod}(A)$. We need to show that $\mathrm{pdim}(U)$ and $\mathrm{idim}(U)$ are both bounded by d. We do this for $\mathrm{pdim}(U)$; the argument for $\mathrm{idim}(U)$ is similar. Let P be a projective resolution of U, with differential π. Set $V = \mathrm{Im}(\pi_d)$; this is a submodule of P_{d-1}. Consider the map $\pi_d : P_d \to V$ as a chain map $P \to V[d]$. Since $0 = \mathrm{Ext}^d_A(U,V) = \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, V[d])$ this chain map is homotopic to zero. Since a homotopy from P to $V[d]$ is zero except possible in degree $d-1$, this is equivalent to the existence of an A-homomorphism $\varphi : P_{d-1} \to V$ satisfying $\pi_d = \varphi \circ \pi_d$.



Thus

$$\varphi \circ \pi_d = \pi_d = \mathrm{Id}_V \circ \pi_d$$

That means that φ restricts to the identity map on $V = \mathrm{Im}(\pi_d)$. Thus V is a direct summand of P_{d-1}, hence P_{d-1}/V is projective. Note that $V = \mathrm{Im}(\pi_{d-1}) = \ker(\pi_{d-1})$, and hence π_{d-1} induces an injective map $\iota : P_{d-1}/V \to P_{d-2}$. It follows that there is a projective resolution of U of the form

$$\cdots \longrightarrow 0 \longrightarrow P_{d-1}/V \xrightarrow{\iota} P_{d-2} \xrightarrow{\pi_{d-2}} \cdots \longrightarrow P_1 \xrightarrow{\pi_1} P_0$$

where ι is the injective map induced by π_{d-1}. Thus $\mathrm{pdim}(U) \leq d$ as required. □

Examples 2.1.20. We have $\mathrm{gldim}(\mathbb{Z}) = 1$ and $\mathrm{gldim}(k[x]) = 1$, where k is a field. More generally, any principal ideal domain which is not a field has global dimension 1. A field k has global dimension 0 because any k module is a k-vector space, hence has a basis, or equivalently, is free. Thus any k-module U is its own projective resolution concentrated in degree 0, together with the identity map Id_U. By earlier calculations, we have $\mathrm{gldim}(k[x]/(x^2)) = \infty$.

Remark 2.1.21. Work of B. Osofsky in the 1970s explores surprising interactions between set theory and global dimensions of rings. For instance, a consequence of Osofsky's work is that the algebra $\mathbb{R}(x,y,z)$ of real rational functions in three variables viewed as a module over the polynomial subalgebra $\mathbb{R}[x,y,z]$ has global dimension 2 if and only if the continuum hypothesis holds. It was conjectured by J. H. C. Whitehead that for A an abelian group, $\mathrm{Ext}^1_{\mathbb{Z}}(A, Z)$ is zero if and only if A is a free abelian group. Work of S. Shelah from the 1970s shows that the truth of this conjecture depends on the set theory being used.

2.2 Derived categories and functors

Associating a projective or injective resolution to a module over an algebra is a functor from the module category to the homotopy category of chain complexes or cochain complexes. We have made use of this fact to define $\mathrm{Ext}^n_A(U, V)$ in a bifunctorial way. The underlying formalities lead to the following definition of derived functors.

Definition 2.2.1. Let A, B be algebras over a commutative ring k. Let $\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ be a functor, and let n be an integer. The *n-th left derived functor*

$$L_n(\mathcal{F}) : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

of $\mathcal{F}$ is the functor which sends an A-module U to the n-th homology $H_n(\mathcal{F}(P_U))$ of the chain complex obtained from applying $\mathcal{F}$ to a projective resolution P_U of U. This functor sends an A-homomorphism $\alpha : U \to V$ to the map $H_n(\mathcal{F}(\varphi))$, where $\varphi : P_U \to P_V$ is a chain map lifting α. The *n-th right derived functor*

$$R^n(\mathcal{F}) : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

is the functor which sends an A-module U to the n-th cohomology $H^n(\mathcal{F}(I_U))$ of the cochain complex obtained from applying $\mathcal{F}$ to a injective resolution I_U of U. This functor sends an A-homomorphism $\alpha : U \to V$ to $H^n(\mathcal{F}(\psi)$, where $\psi : I_U \to I_V$ is a chain map extending α.

These constructions are well-defined. The resolutions P_U, P_V, I_U, I_V are unique up to unique homotopy. Any functor on $\mathrm{Mod}(A)$ extended to $\mathrm{Ch}(\mathrm{Mod}(A))$ sends (co-)chain maps to (co-)chain maps, homotopies to homotopies, and hence homotopic (co-)chain maps to homotopic (co-)chain maps, thereby inducing the same maps in (co-)homology. These construction principles of left and right derived functors extend verbatim to functors between abelian categories having enough projective and injective objects, respectively. Note that $L_n(\mathcal{F})$ and $R^n(\mathcal{F})$ are zero for n negative, since a projective resolution of V and an injective resolution of U are zero in negative degree. We have analogous notions $L^n(\mathcal{G})$ and $R_n(\mathcal{G})$ for $\mathcal{G} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ a contravariant functor instead of the functor $\mathcal{F}$.

We normally consider the left derived functors $L_n(\mathcal{F})$ only if $\mathcal{F}$ is right exact, because then we have a canonical isomorphism of functors $L_0(\mathcal{F}) \cong \mathcal{F}$. Similarly, we consider the right derived functors $R^n(\mathcal{F})$ only when $\mathcal{F}$ is left exact, because then $R^0(\mathcal{F}) \cong \mathcal{F}$.

Proposition 2.2.2. *Let A, B be a algebras over a commutative ring and let $\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ be a k-linear functor.*

(i) *If $\mathcal{F}$ is right exact, then $L_0(\mathcal{F}) = \mathcal{F}$.*

(ii) *If $\mathcal{F}$ is left exact, then $R^0(\mathcal{F}) = \mathcal{F}$.*

(iii) *If $\mathcal{F}$ is exact, then $L_n(\mathcal{F})$ and $R^n(\mathcal{F})$ are zero for all nonzero integers n.*

Proof. Let U be an A-module, and assume that $\mathcal{F}$ is right exact. Applying $\mathcal{F}$ to a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow U \longrightarrow 0$$

yields an exact sequence of B-modules

$$\mathcal{F}(P_1) \longrightarrow \mathcal{F}(P_0) \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

where we use the assumption that $\mathcal{F}$ is right exact. Thus the degree zero homology of the chain complex

$$\cdots \longrightarrow \mathcal{F}(P_2) \longrightarrow \mathcal{F}(P_1) \longrightarrow \mathcal{F}(P_0) \longrightarrow 0$$

is the cokernel of the map $\mathcal{F}(P_1) \to \mathcal{F}(P_0)$, hence canonically isomorphic to $\mathcal{F}(U)$. Statement (i) follows. Statement (ii) is proved analogously. If $\mathcal{F}$ is exact, then applying $\mathcal{F}$ to a projective (resp. injective) resolution of U yields a complex which is exact in all positive degrees, so has zero (co-)homology in all positive degrees. Thus $L_n(\mathcal{F})$ and $R^n(\mathcal{F})$ are zero for all positive n. As mentioned above, $L_n(\mathcal{F})$ and $R^n(\mathcal{F})$ are trivially zero for negative n, whence statement (iii). □

We interpret Ext and Tor in terms of derived functors. Given A-modules U, V and a right A-module W, if we consider the left exact covariant functor $\mathrm{Hom}_A(U, -) : \mathrm{Mod}(A) \to \mathrm{Mod}(k)$, the right exact contravariant functor $\mathrm{Hom}_A(-, V) : \mathrm{Mod}(A) \to \mathrm{Mod}(k)$, the covariant right exact functor $- \otimes_A V : \mathrm{Mod}(A^{\mathrm{op}}) \to \mathrm{Mod}(k)$ and the covariant right exact $W \otimes_A - : \mathrm{Mod}(A)$ to $\mathrm{Mod}(k)$, then rephrasing the terminology introduced in Section 2.1 yields the following.

Proposition 2.2.3. *Let A, B be algebras over a commutative ring k. Let U, V be a A-modules and let W be a right A-module. We have*

$$\mathrm{Ext}^n_A(U, V) = L^n(\mathrm{Hom}_A(-, V))(U) = R^n(\mathrm{Hom}_A(U, -))(V)$$

$$\mathrm{Tor}^A_n(W, V) = L_n(- \otimes_A V)(W) = L_n(W \otimes_A -)(V)$$

The constructions of left and right derived functors involved in this Proposition are unique up to unique isomorphism, which is why we have stated the isomorphisms as equalities. The underlying ideas can be extended significantly to projective resolutions not just of modules but of certain chain complexes. We first extend the notion of projective/injective resolutions.

Definition 2.2.4. Let A be an algebra over a commutative ring. Let X be a bounded below chain complex of A-modules and Y a bounded below cochain complex of A-modules.

(i) A *projective resolution of X* is a pair (P, π) consisting of a bounded below complex P of projective A-modules together with a quasi-isomorphism $\pi : P \to X$.

(ii) An *injective resolution* of Y is a pair (I, ι) consisting of a bounded below cochain complex of injective A-modules and a quasi-isomorphism $\iota : Y \to I$.

While it is easy to show the existence of projective and injective resolutions for A-modules, one needs to work a little more to establish this for bounded below (co-)chain complexes.

Theorem 2.2.5. *Let A be an algebra over a commutative ring.*

(i) Every bounded below chain complex X of A-modules has a projective resolution (P, π).

(ii) Every bounded below cochain complex Y of A-modules has an injective resolution (I, ι).

Proof. We construct a projective resolution(P, π) of X inductively. Let (X, δ) be a bounded below chain complex of A-modules. By an appropriate shift of X, we may assume that $X_n = 0$ for any negative integer n. We set $P_n = 0$ and $\pi_n = 0$ for any negative integer n. For n non negative, we construct P_n and π_n inductively. Assume that for $i < n$ we have already constructed a projective module P_i, a map $\pi_i : P_i \to X_i$ and a map $\epsilon_i : P_i \to P_{i-1}$ with the following properties:

(1) $\pi_{i-1} \circ \epsilon_i = \delta_i \circ \pi_i$, for $i < n$;

(2) the map π_i is surjective and induces an isomorphism $\ker(\epsilon_i)/\mathrm{Im}(\epsilon_{i+1}) \cong H_i(X)$ for $i < n-1$;

(3) π_{n-1} maps $\ker(\epsilon_{n-1})$ onto $\ker(\delta_{n-1})$;

(4) $\ker(\epsilon_{n-1})$ is the inverse image of $\ker(\delta_{n-1})$ in P_{n-1} under π_{n-1}.

We construct P_n and π_n as follows. Let U be the inverse image of $\mathrm{Im}(\delta_n)$ in P_{n-1}. Thus π_{n-1} induces an isomorphism $\ker(\epsilon_{n-1})/U \cong H_{n-1}(X)$. Let V be the submodule of $U \oplus X_n$ consisting of all $(u, x) \in U \oplus X_n$ satisfying $\pi_{n-1}(u) = \delta_n(x)\}$. That is, V is the kernel of the map $(-\pi_{n-1}, \delta_n) : U \oplus X_n \to X_{n-1}$, or equivalently, V is the pullback of the maps $\pi_{n-1}|_U$ and δ_n. Take a projective cover $\pi_V : P_n \to V$ of V. We define $\epsilon_n : P_n \to P_{n-1}$ to be the composition of π_V followed by the projection $V \to P_{n-1}$ mapping $(u, x) \in V$ to u. We define $\pi_n : P_n \to X_n$ to be the composition of π_V followed by the projection $V \to X_n$ mapping $(u, x) \in V$ to x. Then π_n is surjective, by construction. Moreover, we have $\mathrm{Im}(\epsilon_n) = U$, thus π_{n-1} induces an isomorphism $\ker(\epsilon_{n-1})/\mathrm{Im}(\epsilon_n) \cong H_{n-1}(X)$. Also, by the construction of π_n and ϵ_n we have $\pi_{n-1} \circ \epsilon_n = \delta_n \circ \pi_n$. Finally, since $\ker(\epsilon_n)$ is the inverse image in P_n of the submodule $\{(0, x)|x \in \ker(\delta_n)\}$ of V, the map π_n sends $\ker(\epsilon_n)$ onto $\ker(\delta_n)$ and $\ker(\epsilon_n)$ is the inverse image of $\ker(\delta_n)$ in P_n. This concludes the proof of (i). The proof of (ii) is dual to this. □

For projective or injective resolutions of modules we noted that they depend functorially on the module, up to homotopy. The same is true in the more general context of projective or injective resolutions of bounded below (co-)chain complexes.

Theorem 2.2.6. *Let A be an algebra over a commutative ring, let X, Y be bounded below chain complexes of A-modules and let (P_X, π_X), (P_Y, π_Y) be projective resolutions of X, Y, respectively. Let $f : X \to Y$ be a chain map. Then the following hold.*

(i) There is, up to homotopy, a unique chain map $P_f : P_X \to P_Y$ such that $\pi_Y \circ P_f \sim f \circ \pi_X$.

(ii) f is a quasi-isomorphism if and only if P_f is a homotopy equivalence.

(iii) Any two projective resolutions (P_X, π_X) and (P'_X, π'_X) of X are uniquely homotopy equivalent.

(iv) If $f \sim 0$ then $P_f \sim 0$.

Proof. The existence and uniqueness, up to homotopy, of P_f satisfying (i) follows from Theorem 1.3.8. Since π_X, π_Y are quasi-isomorphisms, it follows that f is a quasi-isomorphism if and only if P_f is a quasi-isomorphism. For (ii), suppose that P_f is a quasi-isomorphism. We need to show that P_f is a homotopy equivalence. This can be done directly by constructing a homotopy invere inductively. For a more structural proof, see [7, Theorem 2.18.4]. Applying (i) to $X = Y$ and $f = \mathrm{Id}_X$ yields a chain map $P_{\mathrm{Id}} : P_X \to P'_X$ which is unique up to homotopy subject to the condition $\pi_{X'} \circ P_{\mathrm{Id}} \sim \pi_X$ and a chain map $P'_{\mathrm{Id}} : P'_X \to P_X$ which is unique up to homotopy subject to the condition $\pi_X \circ P'_{\mathrm{Id}} \sim \pi_{X'}$. Together we get that $\pi_X \circ P'_{\mathrm{Id}} \circ P_{\mathrm{Id}} \sim \pi_X$ and $\pi_{X'} \circ P_{\mathrm{Id}} \circ P'_{\mathrm{Id}} \sim \pi'_X$. Now the identity chain maps Id_{P_X} and $\mathrm{Id}_{P'_X}$ on P_X and P'_X also satisfy $\pi_X \circ \mathrm{Id}_{P_X} \sim \pi_X$ and $\pi'_X \circ \mathrm{Id}_{P'_X} \sim \pi'_X$. The uniqueness statement in (i) implies that $P'_X \circ P_X \sim \mathrm{Id}_{P_X}$ and $P_X \circ P'_X \sim \mathrm{Id}_{P'_X}$, whence (iii). Since a homotopy equivalence is a quasi-isomorphism, statement (iv) follows from (ii). □

We have analogous statements for injective resolutions. The key point of this Theorem and its analogue is that it allows us to indentify quasi-isomorphisms between bounded below chain complexes as homotopy equivalences between their projective (resp.) resolutions. In other words, the passage to projective (resp. injective) resolution turns exactly the class of quasi-isomorphisms into isomorphisms in the homotopy category of bounded below chain complexes of projective modules.

Definition 2.2.7. Let A be an algebra over a commutative ring. The *bounded below derived category* $D^-(\mathrm{Mod}(A))$ is the homotopy category $K^-(\mathrm{Proj}(A))$ of bounded below chain complexes of projective A-modules. The *bounded above derived category* $D^+(\mathrm{Mod}(A))$ is the homotopy category $K^+(\mathrm{Inj}(A))$ of cochain complexes of bounded below cochain complexes of injective A-modules. The *bounded derived category* $D^b(\mathrm{Mod}(A))$ is the full subcategory of $D^-(\mathrm{Mod}(A))$ of all bounded below chain complexes of projective A-modules with nonzero homology in at most finitely many degrees.

In other words, $D^-(\mathrm{Mod}(A))$ consists of all projective resolutions of bounded below chain complexes, $D^b(\mathrm{Mod}(A))$ consistes of all projective resolutions of bounded chain complexes, and $D^+(\mathrm{Mod}(A))$ consists of all injective resolutions of bounded below cochain complexes. The injective analogue of $D^b(\mathrm{Mod}(A))$ would be the category of all injective resolutions of bounded cochain complexes of A-modules. Via the passage of rewriting cochain complexes as chain complexes and using results of this section, one can show that this yields a category which is equivalent to $D^b(\mathrm{Mod}(A))$. When working with more general abelian categories one may need the injective version of the derived bounded category - for instance, in the category of sheaves on a topological space there are enough injective objects but not enough projective objects.

The derived categories defined above are solutions to the universal problem of inverting quasi-isomorphisms between bounded below chain complexes. Indeed, the following is simply a reformulation of earlier results.

Theorem 2.2.8. *Let A be an algebra over a commutative ring.*

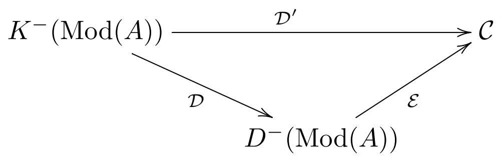
(i) *There is up to unique isomorphism of functors, a unique functor*

$$\mathcal{D} : K^-(\mathrm{Mod}(A)) \to D^-(\mathrm{Mod}(A))$$

sending a bounded below complex X to a projective resolution P_X and sending the homotopy class of a chain map $f : X \to Y$ to the homotopy class of $P_f : P_X \to P_Y$.

(ii) *The functor $\mathcal{D} : K^-(\mathrm{Mod}(A)) \to D^-(\mathrm{Mod}(A))$ is right adjoint to the inclusion functor $D^-(\mathrm{Mod}(A)) \to K^-(\mathrm{Mod}(A))$, and $\mathcal{D}$ restricts to the identity functor on $D^-(\mathrm{Mod}(A))$.*

(iii) *The functor $\mathcal{D}$ is universal for the property of inverting quasi-isomorphisms. More precisely, given an additive functor $\mathcal{D}' : K^-(\mathrm{Mod}(A)) \to \mathcal{C}$ which sends quasi-isomorphisms to isomorphisms, there is a unique functor $\mathcal{E} : D^-(\mathrm{Mod}(A)) \to \mathcal{C}$, up to isomorphism of functors, satisfying $\mathcal{D}' \cong \mathcal{E} \circ \mathcal{D}$.*



Proof. Statement (i) is a reformulation of Theorem 2.2.6. Statement (ii) is a consequence of Theorem 1.3.8: since π_Y is a quasi-isomorphism, it induces an isomorphism

$$\mathrm{Hom}_{D^-(\mathrm{Mod}(A))}(P_X, P_Y) \cong \mathrm{Hom}_{K^-(\mathrm{Mod}(A))}(P_X, Y) \ .$$

The functor $\mathcal{E}$ in statement (iii) is constructed by restricting $\mathcal{D}'$ to $D^-(\mathrm{Mod}(A))$, that is, by setting $\mathcal{E}(P) = \mathcal{D}'(P)$ for any bounded below complex P of projective A-modules. Let (P_X, π_X) be a projective resolution of a bounded below chain complex X, such that $\mathcal{D}(X) = P_X$. Then $\mathcal{E}(P_X) = \mathcal{D}'(P_X)$. The quasi-isomorphism π_X yields a quasi-isomorphism $\mathcal{D}'(P_X) \to \mathcal{D}'(X)$. This shows that the functors $\mathcal{D}'$ and $\mathcal{E} \circ \mathcal{D}$ are isomorphic. □

Derived categories can be constructed in much greater generality for not necessarily bounded chain complexes in an abelian category, modulo some set theoretic precautions.

Chapter 3

Cohomology of algebras, groups, and topological spaces

3.1 Hochschild cohomology

Let A be an algebra over a commutative ring k. In what follows we will work with the category of A-A-bimodules. This category can be identified with the category of modules over the algebra $A \otimes_k A^{\mathrm{op}}$, where A^{op} is the opposite algebra of A. That is, $A^{\mathrm{op}} = A$ as a k-module, with product $a \cdot b = ba$, where the expression $a \cdot b$ is the product in A^{op} and ba is the product in A. More precisely, an A-A-bimodule M can be viewed as an $A \otimes_k A^{\mathrm{op}}$-module via

$$(a \otimes b) \cdot m = amb \ ,$$

where a, $b \in A$, $m \in M$. Given a left $A \otimes_k A^{\mathrm{op}}$-module M, the same equation can be used to define a bimodule structure on M. The advantage of working with $A \otimes_k A^{\mathrm{op}}$-modules is that it makes all the module theoretic machinery available for bimodules - such as projective and injective resolutions, Ext and Tor, for instance. Note that A is itself an A-A-bimodule, hence an $A \otimes A^{\mathrm{op}}$-module, through left and right multiplication by A on itself. The tensor product $A \otimes_k A$ endowed with left multiplcation by A on the first copy of A and right multiplication on the second copy of A becomes in this way an A-A-bimodule. When viewed as an $A \otimes_k A^{\mathrm{op}}$-module, this is then equal to the free $A \otimes_k A^{\mathrm{op}}$-module of rank 1 because the right action of A on the second copy of A is the same as the left action of A^{op} on A^{op}.

Definition 3.1.1. Let A be an algebra over a commutative ring k. Suppose that A is projective as a k-module. Let M be an A-A-bimodule. The *Hochschild cohomology of A with coefficients in M* is the graded k-module

$$HH^*(A; M) = \mathrm{Ext}^*_{A \otimes_k A^{\mathrm{op}}}(A, M)$$

and the *Hochschild cohomology of A* is the graded k-algebra

$$HH^*(A) = HH^*(A; A) = \mathrm{Ext}^*_{A \otimes_k A^{\mathrm{op}}}(A, A)$$

By earlier results, $HH^*(A;M)$ is a graded right $HH^*(A)$-module. In order to calculate Hochschild cohomology, we will need a projective resolution P of A as an $A \otimes_k A^{\mathrm{op}}$-module, to which we then apply the functor $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(-,M)$ and take the cohomology of the resulting cochain complex. Definition 3.1.1 makes sense even if A is not projective as a k-module, and this is in fact the point of view taken in the book by Cartan and Eilenberg [3]. Hochschild's original definition relies on a specific chain complex which also does not require A to be projective as a k-module, but which yields a projective resolution of A as an $A \otimes_k A^{\mathrm{op}}$-module only if A is projective as a k-module; in general, Hochschild's definition yields a version with Ext replaced by a notion of Ext relative to k.

We start by observing that there is a canonical chain complex (X, d) of $A \otimes_k A^{\mathrm{op}}$-modules of the form

$$\cdots \longrightarrow A \otimes_k A \otimes_k A \xrightarrow{d_1} A \otimes_k A \xrightarrow{d_0} A \longrightarrow 0$$

with differential d constructed as follows. Multiplication in A induces a surjective homomorphism of A-A-bimodules

$$d_0 : A \otimes_k A \to A,\ a \otimes b \mapsto ab,\ (a, b \in A)$$

Tensoring this map on the right and left by A and taking the difference of the two resulting maps yields an A-A-bimodule homomorphism

$$d_1 : A \otimes_k A \otimes_k A \to A \otimes_k A\ ,\quad a \otimes b \otimes c \mapsto ab \otimes c - a \otimes bc$$

The sign of the right term ensures that the image of this map is equal to the kernel of the first map given by multiplication in A. We can iterate this construction, and we will show that we obtain in the process a resolution of A as an A-A-bimodule of the form

$$\cdots \longrightarrow A^{\otimes 3} \longrightarrow A^{\otimes 2} \longrightarrow 0$$

together with the quasi-isomorphism given by the multiplication map $A^{\otimes 2} \to A$. Here the notation is

$$A^{\otimes n} = A \otimes_k A \otimes_k A \otimes_k \cdots \otimes_k A\ ,$$

where we tensor $n \geq 1$ copies of A over k. For later use we adopt the convention $A^{\otimes 0} = k$. For $n \geq 1$ we regard $A^{\otimes n}$ as an A-A-bimodule in such a way that A acts on the left by left multiplication on the first copy of A in this tensor product and A acts on the right by right multiplication on the last copy of A. The intermediate copies of A matter for this bimodule structure only as far as their k-module structure is concerned.

Proposition 3.1.2. *Let A be an algebra over a commutative ring k. For $n \geq -1$ set $X_n = A^{\otimes n+2}$ and for $n \geq 0$ denote by $d_n : X_n \to X_{n-1}$ the $A \otimes_k A^{\mathrm{op}}$-homomorphism given by*

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}\ .$$

Set $X_n = 0$ for $n \leq -2$ and $d_n = 0$ for $n \leq -1$. Then $X = (X_n, d_n)_{n\in\mathbb{Z}}$ is an acyclic complex as $A \otimes_k A^{\mathrm{op}}$-modules. More precisely, (X_n, d_n) is contractible as a complex of left A-modules and as a complex of right A-modules.

Proof. For $n \geq -1$ and i satisfying $0 \leq i \leq n$ define the $A \otimes_k A^{\mathrm{op}}$-homomorphism $d_{n,i} : A^{\otimes(n+2)} \to A^{\otimes(n+1)}$ by setting

$$d_{n,i}(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \ .$$

Then $d_n = \sum_{i=0}^{n} (-1)^i d_{n,i}$. The maps $d_{n,i}$ and hence d_n are bimodule homomorphisms. For $n \geq 0$ we have

$$d_{n-1} \circ d_n = \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} d_{n-1,j} \circ d_{n,i} \ .$$

We show that the terms in this sum can be paired with opposite signs. If $j \geq i$, then $d_{n-1,j} \circ d_{n,i} = d_{n-1,i} \circ d_{n,j+1}$. If $j < i$, then $d_{n-1,j} \circ d_{n,i} = d_{n-1,i-1} \circ d_{n,j}$. Thus pairing the summand indexed (i,j) with that indexed by $(j+1,i)$ if $j \geq i$ and with $(j, i-1)$ if $j < i$ shows that all summands cancel. This shows that (X_n, d_n) is a chain complex of $A \otimes A^{\mathrm{op}}$-modules.

Define homomorphisms of right A-modules $h_n : X_n \to X_{n+1}$ by

$$h_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}$$

for $n \geq -1$ and $h_n = 0$ for $n \leq -2$. We will shows that (X_n, d_n) is contractible as a complex of right A-modules with the homotopy h. We need to show that

$$\mathrm{Id}_{X_n} = d_{n+1} \circ h_n + h_{n-1} \circ d_n$$

for all $n \in \mathbb{Z}$. We have

$$(h_{n-1} \circ d_n + d_{n+1} \circ h_n)(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) =$$

$$= h_{n-1}(\sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \ + \ d_{n+1}(1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) =$$

$$= \sum_{i=0}^{n} (-1)^i 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \ +$$

$$a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} + \sum_{i=0}^{n} (-1)^{i+1} 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} =$$

$$= a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \ .$$

This shows that (X_n, d_n) is contractible as a complex of right A-modules. In particular, this complex is acyclic. A similar argument, using a homotopy of left A-modules sending $a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}$ to $a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1$ shows that this complex is also contractible as a complex of left A-modules. □

The complex (X_n, d_n) is not always a projective resolution of A as an $A \otimes_k A^{\mathrm{op}}$-module. This is where we will need the extra condition on A in the definition to ensure this: we will need to assume that A is projective as a k-module. Then $A^{\otimes n}$ is projective as a k-module (because the tensor product of two free k-modules is again free). Thus $A^{\otimes n+2} = A \otimes_k A^{\otimes n} \otimes_k A$ is a projective $A \otimes_k A^{\mathrm{op}}$-module. It follows that the chain complex P with $P_n = X_n = A^{\otimes n+2}$ for $n \geq 0$ with differential $d_n : P_n \to P_{n-1}$ for $n > 0$ together with the quasi-isomorphism $d_0 : P_0 \to A$ given by multiplication in A is a projective resolution of A as an $A \otimes_k A^{\mathrm{op}}$-module. This resolution is called the *bar resolution of* A and can be used to calculate the Hochschild cohomology of A with coefficients in any $A \otimes_k A^{\mathrm{op}}$-module M as

$$HH^n(A; M) = H^n(\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P, M))$$

We will describe the cochain complex $\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(P, M)$ more explicitly; this will in particular yield interpretations of low degree Hochschild cohomology. The term in degree $n \geq 0$ of the cochain complex $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P, M)$ is equal to $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes n+2}, M)$. This can be simplified as follows. We make use of the convention $A^{\otimes 0} = k$.

Lemma 3.1.3. *Let A be an algebra over a commutative ring k. For $n \geq 0$ we have a canonical isomorphism*

$$\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes n+2}, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M) \ .$$

This isomorphism sends an $A \otimes_k A^{\mathrm{op}}$-homomorphism $\zeta : A^{\otimes n+2} \to M$ to the unique k-linear map $\tau : A^{\otimes n} \to M$ defined for $n > 0$ by $\tau(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \zeta(1 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes 1)$ and for $n = 0$ by $\tau(1) = \zeta(1 \otimes 1)$.

Proof. To show that this is an isomorphism, we describe explicitly an inverse map as follows. Let $\tau : A^{\otimes n} \to M$ be a k-linear map. Define $\zeta : A^{\otimes n+2} \to M$ by setting $\zeta(a_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_n\otimes a_{n+1}) = a_0\tau(a_1 \otimes a_2 \otimes \cdots \otimes a_n)a_{n+1}$; this expression is well-defined: the element in the middle belongs to M, and since M can be regarded as a bimodule, we can multiply this element on the left by a_0 and on the right by a_{n+1}. A trivial verification shows that ζ defined this way is an $A \otimes_k A^{\mathrm{op}}$-module homomorphism and that the given assignment is inverse to that described in the statement. □

The above Lemma is a special case of the Tensor-Hom adjunction. In particular, the degree zero term of the complex $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P, M)$ can be identified as

$$\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A \otimes_k A, M) \cong \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(k, M) \cong M \ ,$$

where the first isomorphism is from Lemma 3.1.3, and the second isomorphism sends a linear map $\tau : k \to M$ to $\tau(1)$. The composition of these two isomorphisms sends a bimodule homomorphism $\zeta : A\otimes_k A \to M$ to the element $\zeta(1\otimes 1)$ in M. With this identification, we can describe the cochain complex $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P, M)$ as follows.

Theorem 3.1.4. *Let A be an algebra over a commutative ring k such that A is projective as a k-module. Let M be an $A \otimes_k A^{\mathrm{op}}$-module. Define k-modules $C^n(A; M)$ for $n \geq 0$ by setting*

$$C^n(A, M) = \mathrm{Hom}_k(A^{\otimes n}, M)$$

Define maps $\delta^n : \mathrm{Hom}_k(A^{\otimes n}; M) \to \mathrm{Hom}_k(A^{\otimes n+1}; M)$ for $n \geq 0$ by by setting

$$\delta^n(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_n) =$$

$$= a_0 f(a_1 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n} (-1)^i f(a_0 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_n) + (-1)^{n+1} f(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})a_n$$

for any $f \in Hom_k(A^{\otimes n}, M)$. Then $(C^n(A; M), \delta^n)$ is a cochain complex which is isomorphic to $\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(P; M)$, where P is the bar resolution of A as before. In particular, the cohomology of this cochain complex is the Hochschild cohomology $HH^*(A)$ of A.

Proof. By Lemma 3.1.3, $C^n(A;M)$ is isomorphic to the term in degree n of $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P;M)$. One needs to chase the differential through these isomorphisms; this yields the maps as described. For $n \geq 0$, we have from 3.1.3 an isomorphism

$$\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes n+2}, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M) \ .$$

Let $f \in \mathrm{Hom}_k(A^{\otimes n}, M)$. Through the previous isomorphism, this corresponds to the homomorphism $\hat{f} \in \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes n+2}, M)$ given by the formula

$$\hat{f}(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 f(a_1 \otimes \cdots \otimes a_n)a_{n+1} \ .$$

By the definition of δ^n, we have $\delta^n(\hat{f}) = \hat{f} \circ \delta_{n+1} \in \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes n+3}; M)$, and $\delta^n(f)$ is then obtained from restricting $\hat{f} \circ \delta_{n+1}$ to $1 \otimes A^{\otimes n+1} \otimes 1$. So we need to calculate

$$(\hat{f} \circ \delta_{n+1})(1 \otimes a_0 \otimes \cdots \otimes a_{n+1} \otimes 1)$$

and this is the expression as in the statement of the theorem. Note that in the expression $1 \otimes a_0 \otimes \cdots \otimes a_{n+1} \otimes 1$ the term a_i is now in the component $i+1$ because of the first component equal to 1; this is the reason why in the sum, the term indexed by i multiplies $a_{i-1}a_i$ and not $a_i a_{i+1}$. □

Remark 3.1.5. With the notation of the previous theorem, we have $C^0(A;M) = \mathrm{Hom}_k(k, m) \cong M$, because a k-linear map $\tau : k \to M$ is uniquely determined by its value $\tau(1_k)$. If we identify $C^0(k;M)$, then the differential $\delta^0 : M \to \mathrm{Hom}_k(A, M)$ sends $m \in M$ to the map $\delta^0(m) : A \to M$ given by $\delta^0(m)(a) = am - ma$. Thus the cochain complex calculating Hochschild cohomology takes the form

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\delta^0} \mathrm{Hom}_k(A, M) \xrightarrow{\delta^1} \mathrm{Hom}_k(A \otimes_k A, M) \longrightarrow \cdots$$

We can use this cochain complex to calculate Hochschild cohomology in low degrees. For M an A-A-bimodule we set $M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}$. Note that M^A is a left and right $Z(A)$-submodule of M and that $Z(A) = A^A$.

Proposition 3.1.6. *Let A be an algebra over a commutative ring k such that A is projective as a k-module. We have a canonical isomorphism $HH^0(A;M) = M^A$. In particular, we have $HH^0(A) \cong Z(A)$.*

Proof. With the notation of 3.1.4 and the identification $C^0(A;M) = M$, we have

$$HH^0(A;M) = \ker(\delta^0) = \{m \in M \mid am - ma = 0 \text{ for all } a \in A\} = M^A \ .$$

Since $A^A = Z(A)$, the second statement follows. □

The Hochschild cohomology in degree 1 has an interpretation in terms of derivations.

Definition 3.1.7. Let A be an algebra over a commutative ring k. For M an A-A-bimodule, a k-linear map $f : A \to M$ is called a *derivation* if $f(ab) = af(b) + f(a)b$ for all $a, b \in A$.

The set $\mathrm{Der}(A;M)$ of derivations $A \to M$ is a k-subspace of $\mathrm{Hom}_k(A, M)$.

Exercise 3.1.8. Let A be an algebra over a commutative ring k. Let M be an A-A-bimodule. Show that for a fixed element $m \in M$, the map $[-, m]$ sending $a \in A$ to the additive commutator $[a, m] = am - ma$ is a derivation.

Definition 3.1.9. Let A be an algebra over a commutative ring. Any derivation $A \to M$ which is equal to $[-, m]$ for some $m \in M$ is called an *inner derivation.*

The set $\mathrm{IDer}(A)$ of inner derivations from A to M is a subspace of $\mathrm{Der}(A; M)$. If $A = M$ we set $\mathrm{IDer}(A) = \mathrm{IDer}(A; A)$ and $\mathrm{Der}(A) = \mathrm{Der}(A; A)$.

Proposition 3.1.10. *Let A be an algebra over a commutative ring k such that A is projective as a k-module. We have a canonical isomorphism $HH^1(A; M) \cong \mathrm{Der}(A; M)/\mathrm{IDer}(A; M)$.*

Proof. With the notation of 3.1.4, we have

$$HH^1(A; M) = \ker(\delta^1)/\mathrm{Im}(\delta^0) \ .$$

It suffices to verify that

$$\ker(\delta^1) = \mathrm{Der}(A; M)$$

and

$$\mathrm{Im}(\delta^0) = \mathrm{IDer}(A; M) \ .$$

Let $f \in \mathrm{Hom}_k(A; M)$. Then $\delta^1(f) \in \mathrm{Hom}_k(A \otimes_k A; M)$ is defined by $\delta^1(f)(a \otimes b) = af(b) - f(ab) + f(a)b$. Thus f belongs to $\ker(\delta^1)$ if and only if

$$f(ab) = af(b) + f(a)b$$

for all a, b in A, hence if and only if f is a derivation. This shows that $\ker(\delta^1) = \mathrm{Der}(A; M)$. We have $f \in \mathrm{Im}(\delta^0)$ if and only if there exists $m \in M$ such that $f = \delta^0(m)$, that is, if and only if $f(a) = am - ma$ for all $a \in A$, which is equivalent to $f \in \mathrm{IDer}(A; M)$. This shows $\mathrm{Im}(\delta^0) = \mathrm{IDer}(A; M)$, whence the result. □

Exercise 3.1.11. Let A be an algebra over a commutative ring k. Let f, $g : A \to A$ be derivations. Show that $[f, g] = f \circ g - g \circ f$ is a derivation. Show that if $c \in A$ and $g = [-, c]$, then $[f, g] = [-, f(c)]$.

Remark 3.1.12. There is a lot more structure on Hochschild cohomology. The previous exercise shows that the space of derivations $\mathrm{Der}(A)$ on A is a Lie subalgebra of $\mathrm{End}_k(A)$ with the bracket $[f, g] = f \circ g - g \circ f$, and that $\mathrm{IDer}(A)$ is an ideal in the Lie algebra $\mathrm{Der}(A)$. Thus $HH^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A)$ is a Lie algebra. It was shown by Gerstenhaber that this Lie algebra structure extends to a graded Lie algebra structure of degree -1 on $HH^*(A)$ If A is a finite-dimensional algebra over $\mathbb{C}$, then its outer automorphism group $\mathrm{Out}(A)$ is an algebraic group. The tangent space of the connected component of this algebraic group is the Lie algebra $HH^1(A)$. See for instance Keller [5] for an application and references for this point of view. For an algebra A over an algebraically closed field of prime characteristic p it is still true that $\mathrm{Out}(A)$ is an algebraic group, but its tangent space need not be $HH^1(A)$.

One can use any projective resolution P of A as an $A \otimes_k A^{\mathrm{op}}$-module to obtain projective resolutions of an arbitrary A-module U by tensoring with U over A.

Proposition 3.1.13. *Let A be an algebra over a commutative ring k such that A is projective as a k-module. Let (P, μ) be a projective resolution of A as an $A \otimes_k A^{\mathrm{op}}$-module. Regard P as a complex of A-A-bimodules. Let U be an A-module which is projective as a k-module. Then $(P \otimes_A U, \mu \otimes \mathrm{Id}_U)$ is a projective resolution of U. The map sending a chain map $\zeta : P \to P[n]$ to the chain map $\zeta \otimes \mathrm{Id}_U : P \otimes_A U \to (P \otimes U)[n]$ induces a homomorphism of graded algebras*

$$HH^*(A) \to \mathrm{Ext}_A(U, U)$$

Proof. We first note that the statement makes sense. Applying the functor $- \otimes_A U$ to P yields a chain complex $P \otimes_A U$. The terms of P are projective as bimodules, hence direct summands of free bimodules, so direct summands of sums of copies of the free rank one bimodule $A \otimes_k A$. Thus the terms of $P \otimes_A U$ are direct summands of sums of copies of the left A-module $A \otimes_k U$. Now U is projective as a k-module, to isomorphic to a direct summand of a free k-module, hence a direct summand of a sum of copies of k. Thus, as a left A-module, $A \otimes_k U$ is a direct summand of a sum of copies of $A \otimes_k k \cong A$. This shows that the terms of $P \otimes_k U$ are all projective as A-modules. The quasi-isomorphism μ is the surjective $A \otimes_k A^{\mathrm{op}}$-homomorphism $A \otimes_k A \to A$ given by multiplication in A. Thus $\mu \otimes \mathrm{Id}_U$ is a surjective A-homomorphism $A \otimes_k A \otimes_A U \to A \otimes_A U$. After identifying $A \otimes_A U = U$, this is the surjective A-homomorphism $\nu : A \otimes_k U \to U$ sending $a \otimes u$ to au, where $a \in A$ and $u \in U$. If we restrict attention to the right A-module structure on P, then (P, μ) remains a projective resolution of A as a right A-module. Since A is projective (even free of rank 1) as a right A-module, A itself with the identity Id_A is its own projective resolution as a right A-module. Since projective resolutions are unique up to homotopy, it follows that then map μ, when considered as a homomorphism of right A-modules, induces a homotopy equivalence $P \simeq A$ as chain complexes of right A-modules. But then tensoring by $- \otimes U$ implies that ν induces a homotopy equivalence $P \otimes_A U \simeq A \otimes_A U \cong U$ as complexes of k-modules. Bringing back the left A-module structure, this shows that ν is indeed a quasi-isomorphism $P \otimes_A U \to U$ as required. □

Let A be an algebra over a commutative ring k such that A is projective as a k-module. Let M be an A-A-bimodule. The k-module $HH^2(A; M)$ parametrises associative algebra structures on $A \oplus M$ such that the canonical projection $A \oplus M \to A$ is an algebra homomorphism and such that M becomes an ideal which squares to zero. That is, the multiplication in $A \oplus M$ is given by

$$(a, m)(b, n) = (ab, an + mb + \alpha(a, b))$$

where a, $b \in A$, m, $n \in M$ and $\alpha(a, b) \in M$. This defines a bilinear map $\alpha : A \times A \to M$. A short verification shows that the associativity of this multiplication is equivalent to

$$\alpha(a, b)c + \alpha(ab, c) = a\alpha(b, c) + \alpha(a, bc)$$

for all a, b, $c \in A$. If we extend α to the unique linear map $\alpha : A \otimes_k A \to M$ and bring all terms in the previous equality to one side, then this reads

$$a\alpha(b \otimes c) - \alpha(ab \otimes c) + \alpha(a \otimes bc) - \alpha(a \otimes b)c = 0$$

which is equivalent to $\alpha \in \ker(\delta^2)$. Thus a linear map α yields an associative multiplication on $A \oplus M$ as above if and only if $\alpha \in \ker(\delta^2)$. We denote this algebra by $T_\alpha(A; M)$.

Exercise 3.1.14. With the notation above, let α, $\alpha' \in \ker(\delta^2)$. Show that there is an isomorphism of algebras

$$T_\alpha(A \oplus M) \cong T_{\alpha'}(A \oplus M)$$

which induces the identity on the ideals M and on the quotients A if and only if α and α' determine the same class in $HH^2(A; M)$.

The zero class in $HH^2(A; M)$ corresponds to what is called the *trivial extension algebra*

$$T(A \oplus M) = A \oplus M,$$

with multiplication given by $(a, m)(b, n) = (ab, an + mb)$ for all a, $b \in A$ and m, $n \in M$.

Hochschild cohomology is *graded-commutative*; that is, homogeneous elements commute with respect to the cup product up to signs determined by their degrees. In general, Ext-algebras of modules need not be graded-commutative.

Theorem 3.1.15 (Gerstenhaber)**.** *Let A be an algebra over a commutative ring k such that A is projective as a k-module. The algebra $HH^*(A)$ is graded-commutative; that is, for integers m, $n \geq 0$ and $\zeta \in HH^m(A)$, $\eta \in HH^n(A)$, we have $\eta\zeta = (-1)^{mn}\zeta\eta$.*

Thus if one of m, n is even, then η and ζ commute, and if m is odd, then $\zeta^2 = -\zeta^2$, so $\zeta^2 = 0$ unless A is an algebra over a field of characteristic 2. In particular, the even part $HH^{\mathrm{ev}}(A) = \oplus_{i \geq 0} HH^{2i}(A)$ of Hochschild cohomology is strictly commutative, and if A is an algebra over a field of characteristic 2, then $HH^*(A)$ is commutative.

3.2 Cohomology of groups

Let G be a group and k a commutative ring. The *group algebra* kG of G over k is the algebra which is the free k-module having the set of elements of G as a k-basis, endowed with the unique k-bilinear multiplication induced by the group multiplication of G. More explicitly, the elements of kG are the formal sums $\sum_{x\in G} \lambda_x x$, where $\lambda_x \in k$ for all $x \in G$, with only finitely many of the coefficients λ_x nonzero. The sum in kG is given componentwise by the formula

$$(\sum_{x\in G} \lambda_x x) + (\sum_{x\in G} \mu_x x) = \sum_{x\in G} (\lambda_x + \mu_x)x \ ,$$

the scalar multiplication in kG is given by

$$\lambda(\sum_{x\in G} \lambda_x x) = \sum_{x\in G} (\lambda\lambda_x)x,$$

and the product in kG is given by the formula

$$(\sum_{x\in G} \lambda_x x)(\sum_{x\in G} \mu_x x) = \sum_{x,y\in G} \lambda_x\mu_y xy = \sum_{z\in G} (\sum_{x,y\in G, xy=z} \lambda_x\mu_y)z \ ,$$

where as before the coefficients $\lambda_x, \mu_x, \lambda$ are in k, with only finitely many of the λ_x and the μ_x nonzero. The unit element 1_{kG} of the algebra kG is the image in kG of the unit element 1_G of the group G. The images in kG of the elements of the group G become invertible in the algebra kG in such a way that the image in kG of the inverse x^{-1} in G of an element $x \in G$ is the inverse of the image of x in kG. We tend not to notationally distinguish the elements of G from their images in kG unless this is needed to avoid confusion. The associativity of the product in G implies that the multiplication in kG is associative.

The ring k has a trivial kG-module structure, with all group elements acting as identity. This does not mean that all elements of kG act as identity: the action of an element $\sum_{x\in G} \lambda_x x$ on k is given by multiplication with the scalar $\sum_{x\in G} \lambda_x$. This is well-defined, as only finitely many of the λ_x are nonzero. We call k endowed with this module structure the *trivial kG-module*, and denote it again by k, if no confusion arises. The structural homomorphism $\eta : kG \to k$ determined by the trivial module sends an element $\sum_{x\in G} \lambda_x x$ in kG to the scalar $\sum_{x\in G} \lambda_x$ in k. This is a surjective algebra homomorphism, called the *augmentation homomorphism.* Its kernel, denoted $I(kG)$, is the *augmentation ideal in kG.*

Definition 3.2.1. Let G be a group and k a commutative ring. The cohomology in degree $n \geq 0$ of G with coefficients in a kG-module M is defined as

$$H^n(G; M) = \mathrm{Ext}^n_{kG}(k, M)$$

The cohomology in degree $n \geq 0$ of G with coefficients in an abelian group A is defined as

$$H^n(G; A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}; A)$$

More explicitly,

$$H^n(G, M) = H^n(\mathrm{Hom}_{kG}(P, M)),$$

where P is a projective resolution of the trivial kG-module k. Thus, in order to calculate group cohomology, we need to describe a projective resolution of the trivial kG-module k. As in the case of Hochschild cohomology, there is a canonical projective resolution, called the *bar resolution of the trivial kG-module*. Applying the functor $\mathrm{Hom}_{kG}(-, M)$ to this resolution yields a cochain complex whose cohomology is then cohomology of G with coefficients in M. Hochschild cohomology offers a shortcut to this programme. Thanks to Proposition 3.1.13, tensoring the bimodule bar resolution of kG by $- \otimes_{kG} k$ yields a projective resolution of k. This can be described explicitly as follows.

Theorem 3.2.2. *Let G be a group and k a commutative ring. Let M be a kG-module which is projective as a k-module. For $n \geq 0$ set*

$$C^n(G; M) = \{\alpha : G^n \to M\}$$

where G^n is the direct product of n copies of G, with the convention $C^0(G; M) = M$. For $n \geq 0$ define a k-linear map

$$\delta^n : C^n(G; M) \to C^{n+1}(G; M)$$

by setting

$$\delta^n(\alpha)(x_0, x_1, \ldots, x_n) =$$

$$x_0\alpha(x_1, \ldots, x_n) + \sum_{i=1}^{n}(-1)^i\alpha(x_0, .., x_{i-1}x_i, .., x_n) + (-1)^{n+1}\alpha(x_0, .., x_{n-1})$$

with the convention $\delta^0(m)(x) = xm - m$ Here x and the x_i are elements in G.

Proof. The multiplication map $\mu : kG \otimes_k kG \to kG$ has the property that upon tensoring it with $- \otimes_{kG} k$, it yields the augmentation map. Indeed, after identifying $kG \otimes_{kG} k = k$, we get that $\mu \otimes \mathrm{Id}_{kG} : kG \otimes_k k = kG \to k$ is equal to the augmentation map η. Consider the bar resolution P of kG. Applying $- \otimes_{kG} k$ yields a projective resolution $P \otimes_{kG} k$ of the trivial kG-module k. The term in degree n of this resolution is $\mathrm{Hom}_{kG}((kG)^{\otimes n+1}, M)$, where we use the identification $kG \otimes_{kG} k = k$. Just as in Lemma 3.1.3, we have an isomorphism

$$\mathrm{Hom}_{kG}((kG)^{\otimes n+1}, M) \cong \mathrm{Hom}_k((kG)^{\otimes n}, M)$$

where the passage from the right side gto the left side sends a linear map $\tau : (kG)^{\otimes n} \to M$ to the kG-homomrophism $\alpha : (kG)^{\otimes n+1} \to M$ given by $\alpha(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = x_0\tau(x_1 \otimes \cdots \otimes x_n)$. The space $\mathrm{Hom}_k((kG)^{\otimes n}, M)$ is clearly isomorphic to $C^n(G; M)$, since any map $\alpha : G^n \to M$ extends uniquely to a linear map $(kG)^{\otimes n} \to M$. Through these identifications, the differential is as stated in the theorem. □

The last term $\alpha(x_0, .., x_{n-1})$ in the differential of 3.2.2 is seemingly different from the corresponding last term in the differential of the Hochschild cohomology $f(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})a_n$ in 3.1.4. This is because $x \otimes 1$ and $1 \otimes 1$ are equal in $kG \otimes_k k$ for any $x \in G$, and hence the right multiplication by a_n in 3.1.4 gets 'absorbed' by tensoring with $- \otimes_{kG} k$.

Remark 3.2.3. Theorem 3.2.2 describes group cohomology $H^*(G; M)$ as the cohomology of a cochain complex of the form

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\delta^0} C^1(G; M) \xrightarrow{\delta^1} C^2(G; M) \longrightarrow \cdots$$

The first few differentials are explicitly given by

$$\delta^0(m)(x) = xm - m,$$

$$\delta^1(\beta)(x,y) = x\beta(y) - \beta(xy) + \beta(x),$$

$$\delta^2(\alpha)(x,y,z) = x\alpha(y,z) - \alpha(xy,z) + \alpha(x,yz) - \alpha(x,y),$$

where x, y, $z \in G$, $m \in M$, and $\beta : G \to M$ and $\alpha : G \times G \to M$ are maps. This allows us to calculate low degree group cohomology and give it interpretations.

Exercise 3.2.4. Let G be a group, k a commutative ring, and M a kG-module. Show that

$$H^0(G; M) = M^G = \{m \in M \mid xm = m \text{ for all } x \in G\}$$

$$H^1(G; M) = \ker(\delta^1)/\mathrm{Im}(\delta^0)$$

where

$$\ker(\delta^1) = \{\beta : G \to M \mid \beta(xy) = x\beta(y) + \beta(x) \text{ for all } x, y \in G\}$$

$$\mathrm{Im}(\delta^0) = \{\beta : G \to M \mid \text{there exists } m \in M \text{ such that} \beta(x) = xm - m \text{ for all } x \in G\}$$

In particular, show that if G acts trivially on M, then $H^1(G, M) = \mathrm{Hom}(G, M)$, the set of all group homomorphisms from G to the additive group $(M, +)$; that is, the group of all maps $\beta : G \to M$ satisfying $\beta(xy) = \beta(x) + \beta(y)$ for all x, $y \in G$.

Let A be an abelian group acted upon by a group G. That is, A is a $\mathbb{Z}G$-module. We write A and G both multiplicatively, and we write the action of $x \in G$ on $a \in A$ by ${}^x a$. Any short exact sequence of groups of the form

$$1 \longrightarrow A \longrightarrow H \xrightarrow{f} G \longrightarrow 1$$

gives rise to an action of G on A as follows. For $x \in G$, choose an element $\hat{x} \in H$ such that $f(\hat{x}) = x$. Note that $\hat{x}$ is unique up to multiplication by an element in A. Since A is abelian, the conjugation action of $a \in A$ on A is trivial, and hence the conjugation action of $\hat{x}$ and $\hat{x}a$ on A is the same. That is, we have a well-defined action of G on A by setting ${}^x a = \hat{x}a\hat{x}^{-1}$. Let now x, $y \in G$. Then $\widehat{xy}$ and $\hat{x}\hat{y}$ are two elements which satisfy $f(\widehat{xy}) = xy = f(\hat{x}\hat{y})$, and therefore there is an element $\alpha(x, y) \in A$ such that

$$\hat{x}\hat{y} = \widehat{xy}\alpha(x,y) \ .$$

Using the associativity $(\hat{x}\hat{y})\hat{z} = \hat{x}(\hat{y}\hat{z})$ for all x, y, $z \in G$, a short calculation shows that

$$\alpha(x,y)\alpha(xy,z) = {}^x\alpha(y,z)\alpha(x,yz)$$

and that is exactly saying that $\alpha \in \ker(\delta^2)$; that is, α represents a class in $H^2(G; A)$. For a given fixed action of G on A, we consider the set of equivalence classes of group extensions of the form

$$1 \longrightarrow A \longrightarrow H \xrightarrow{f} G \longrightarrow 1$$

with two such extensions

$$1 \longrightarrow A \longrightarrow H \xrightarrow{f} G \longrightarrow 1$$

$$1 \longrightarrow A \longrightarrow H' \xrightarrow{f'} G \longrightarrow 1$$

being equivalent if there is a group isomorphism $\varphi : H \cong H'$ making the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & H & \xrightarrow{f} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{Id}_A} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{Id}_G} & & \\ 1 & \longrightarrow & A & \longrightarrow & H' & \xrightarrow{f'} & G & \longrightarrow & 1 \end{array}$$

commutative.

Exercise 3.2.5. With the notation above, show that there is a bijection between the equivalence classes of group extensions

$$1 \longrightarrow A \longrightarrow H \xrightarrow{f} G \longrightarrow 1$$

of G by A for a given action of G on A and the set of classes $H^2(G; A)$. Show that the zero class in $H^2(G; A)$ corresponds to the split extension where $H = A \rtimes G$.

Definition 3.2.6. Let G be a finite group. The *Schur multiplier of* G is the abelian group $H^2(G; \mathbb{C}^\times) = \mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}; \mathbb{C}^\times)$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is the the multiplicative group of nonzero complex numbers considered with the trivial action of G.

The Schur multiplier $H^2(G; \mathbb{C}^\times)$ of a finite group G parametrises central extensions of G. Moreover, $H^2(G; \mathbb{C}^\times)$ is a finite abelian group. Slightly more precisely, we have the following result.

Theorem 3.2.7. *Let G be a finite group and k an algebraically closed field. Then the abelian group $H^2(G; k^\times)$ is finite, where we consider the multiplicative group $k^\times = k \setminus \{0\}$ with G acting trivially. Moreover, $|G|$ annihilates $H^2(G; k^\times)$.*

Proof. From the description of group cohomology in Theorem 3.2.2 written multiplicatively, we get that $H^2(G; k^\times) = Z/B$, where

$$Z = \{\alpha : G \times G \to k^\times \mid \alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z) \text{ for all } x, y, z \in G\}$$

$$B = \{\alpha : G \times G \to k^\times \mid \text{there exists } \beta : G \to k^\times \text{ such that } \alpha(x, y) = \beta(x)\beta(y)\beta(xy)^{-1} \text{ for all } x, y \in G\}$$

Let $\alpha \in Z$. Define a map $\mu : G \to k^\times$ by setting

$$\mu(x) = \prod_{y \in G} \alpha(x, y)$$

for all $x \in G$. For x, y, $z \in G$, we have the 2-cocycle identity

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z) .$$

Taking the product over all $z \in G$ yields the identity

$$\mu(xy)\alpha(x, y)^{|G|} = \mu(x)\mu(y) .$$

Thus

$$\alpha(x,y)^{|G|} = \mu(x)\mu(y)\mu(xy)^{-1}$$

showing that $\alpha^{|G|} \in B$, and hence $|G|$ annihilates $Z/B = H^2(G;k^\times)$. Since k is algebraically closed, every element in $k^\times$ has a $|G|$-th root, and hence there is a map $\delta : G \to k^\times$ such that $\delta(x)^{|G|} = \mu(x)$ for all $x \in G$. Define a map $\beta : G \times G \to k^\times$ by setting

$$\beta(x,y) = \alpha(x,y)\delta(x)^{-1}\delta(y)^{-1}\delta(xy)$$

for all x, $y \in G$. Then α and β represent the same class in $H^2(G;k^\times)$ because they differ by an element in the subgroup B of Z. And now we also have $\beta(x,y)^{|G|} = 1$ for all x, $y \in G$. Thus β is a map from $G \times G$ to the finite subgroup of $|G|$-th roots of unity in $k^\times$, so there are only finitely many such maps β. Thus $H^2(G;k^\times)$ is a quotient of the finite subgroup

$$U = \{\beta : G \times G \to k^\times \mid \beta(x,y)^{|G|} = 1 \text{ for all } x, y \in G\}$$

of Z, and the result follows. □

Proposition 3.2.8. *Let G be a finite group and k a commutative ring. The functor $- \otimes_{kG} k$ induces a homormophism of graded algebras $HH^*(kG) \to H^*(G;k)$ which is split surjective.*

Proof. The splitting is constructed as follows: consider the diagonal subgroup $\Delta G = \{(x,x) \mid x \in G\}$ of $G \times G$. Then the induced module $\mathrm{Ind}_{\Delta G}^{G\times G}(k) = k(G\times G) \otimes_{k\Delta G} k$ is isomorphic to kG viewed as a $k(G\times G)$ module with $(x,y) \in G \times G$ acting on $z \in kG$ by xzy^{-1}. Through the isomorphism $k(G \times G) \cong kG \otimes_k (kG)^{\mathrm{op}}$ sending (x,y) to $x \otimes y^{-1}$, this is kG viewed as a kG-kG-bimodule. The induction functor $\mathrm{Ind}_{\Delta G}^{G\times G}$ is exact and maps projective kG-modules to projective $k(G \times G)$-modules, hence sends a projective resolution of the trivial kG-module to a projective resolution of the bimodule kG. It follows that this functor induces a map $H^*(G;k) \to HH^*(kG)$, and one checks that this is a splitting as stated. □

Theorem 3.2.9. *Let G be a finite group and k a commutative ring. We have a canonical isomorphism of graded k-modules*

$$HH^*(kG) = \oplus_x \; H^*(C_G(x);k) \ ,$$

where x runs over a set of representatives of the conjugacy classes in G.

This isomorphism is not an isomorphism as graded algebras. The product in $HH^*(kG)$ can be expressed explicitly in terms of the cohomology rings of centralisers of elements; this is due to Siegel and Witherspoon [11].

Remark 3.2.10. Using the above additive decomposition of $HH^*(kG)$ and the classification of finite simple groups, one can show that $HH^1(kG)$ is nonzero whenever k is a field of prime characteristic dividing the order of G.

3.3 Singular cohomology of topological spaces

Definition 3.3.1. For n a non negative integer, the *standard topological n-simplex* is the compact subspace of $\mathbb{R}^{n+1}$ defined by

$$\Delta_n = \{(x_0, x_1, .., x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum_{i=0}^{n} x_i = 1\}$$

For $0 \leq i \leq n$ the *i-th face map* is the map

$$d_i : \Delta_{n-1} \to \Delta_n$$

sending $(x_0, x_1, .., x_{n-1}) \in \Delta_{n-1}$ to $(x_0, x_1, .., x_{i-1}, 0, x_i, .., x_{n-1}) \in \Delta_n$.

Thus if X is a topological space and n a positive integer, then precomposing with any of the $n+1$ different face maps sends any continuous map $\Delta_n \to X$ to a continuous map $\Delta_{n-1} \to X$. Assembling these maps yields singular homology.

Definition 3.3.2. Let X be a topological space. For $n \geq 0$ denote by $S_n(X)$ the free $\mathbb{Z}$-module having as a basis the set of all continuous maps $f : \Delta_n \to X$. For $n \geq 1$ define a k-linear map $\delta_n : S_n(X) \to S_{n-1}(X)$ by setting

$$\delta_n(f) = \sum_{i=0}^{n-1} (-1)^i f \circ d_i$$

for any continuous map $f : \Delta_n \to X$, where $d_i : \Delta_{n-1} \to \Delta_n$ is the face map defined above. We set $\delta_0 = 0$ for notational convenience. One checks that $\delta_n \circ \delta_{n+1} = 0$. We denote by $S_*(X)$ the chain complex thus obtained. The n-th *singular homology of X* is the $\mathbb{Z}$-module

$$H_n(X) = \ker(\delta_n)/\mathrm{Im}(\delta_{n+1})$$

For A an abelian group, we set $S^n(X; A) = \mathrm{Hom}_{\mathbb{Z}}(S_n(X), A)$ and denote by $\delta^n : S^n(X; A) \to S^{n+1}(X; A)$ the map induced by precomposition with δ_{n+1}, with the notational conovention $\delta^{-1} = 0$. The n-th *singular cohomology of X with coefficients in A* is the $\mathbb{Z}$-module

$$H^n(X; k) = \ker(\delta^n)/\mathrm{Im}(\delta^{n-1})$$

Exercise 3.3.3. Let $X = \{*\}$ be a single point. Show that $S_n(X) = \mathbb{Z}$ for $n \geq 0$, and that $\delta_n = \mathrm{Id}_{\mathbb{Z}}$ for n even and $\delta_n = 0$ for n odd. Show that there is a chain homotopy equivalence $S_*(X) \simeq \mathbb{Z}$, where $\mathbb{Z}$ is regarded as the chain complex concentrated in degree 0. Deduce that $H_n(X)$ is zero for $n > 0$ and $\mathbb{Z}$ for $n = 0$.

Definition 3.3.4. Let X, Y be topological spaces and let f, $g : X \to Y$ be continuous maps. We say that f and g are homotopic and write $f \sim g$ if there is a continuous map $F : [0, 1] \times X \to Y$, such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in X$.

A continuous map $F : X \to Y$ is called a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$. In that case we say that X and Y are homotopy equivalent, and we write $X \simeq Y$. A space X is called *contractible* if it is homotopy equivalent to a point $\{*\}$. The following two theorems, stated here without proof (which can be found in many standard sources on algebraic topology), collect some of the fundamental properties of singular homology.

Theorem 3.3.5.

(1) *Singular homology is functorial; that is, for $n \geq 0$, any continuous map $f : X \to Y$ induces a canonical chain map $S(f) : S_*(X) \to S_*(Y)$ by composition with f and hence a map $H_n(f) : H_n(X) \to H_n(Y)$ both of which are functorial.*

(2) *Singular homology is compatible with homotopy; that is, if $f, g : X \to Y$ are homotopic continuous maps, then the induced chain maps $S(f), S(g) : S_*(X) \to S_*(Y)$ are homotopic chain maps, and we have $H_n(f) = H_n(g)$ for all $n \geq 0$.*

(3) *If $f : X \to Y$ is a homotopy equivalence, then $S(f) : S_*(X) \to S_*(Y)$ is a chain homotopy equivalence. In particular, if X is contractible, then $S_*(X) \simeq \mathbb{Z}$.*

For X a topological space and A a subspace, we denote by A° the set of interior points in A; this is the set of all elements $a \in A$ which have an open neighbourhood in X which is contained in A.

Theorem 3.3.6 (Mayer-Vietoris)**.** *Let X be a topological space and A, B subspaces of X such that $A^\circ \cup B^\circ = X$. There is a long exact singular homology sequence*

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots$$

ending at the map $H_0(X) \to 0$.

Given a sphere $S^n = \{(x_0, x_1, .., x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = 1\}$, we can use the above results to calculate its cohomology.

Theorem 3.3.7. *For $n \geq 0$ and $i > 0$ such that $i \neq n$ we have $H_i(S^n) = 0$. For $n > 0$ we have $H_n(S^n) \cong H_0(S^n) \cong \mathbb{Z}$, and we have $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Since $S^0 = \{-1, 1\}$ is the disjoint unsion of two one-point spaces, we have $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_n(S^0) =)$ for $n > 0$. Whenever we puncture a sphere S^n, $n > 0$, by removing a single point, we obtain a contractible space. Denote by A_n the subspace of S^n obtained by removing the 'south pole' $(0, 0, .., 0, -1)$ and by B_n the space obtained by removing the 'north pole' $(0, 0, .., 0, 1)$ of S^n. For $n = 1$, the Mayer-Vietoris sequence takes the following form

$$0 \longrightarrow H_1(S^1) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(S^1) \longrightarrow 0$$

The map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is easily seen to be nonzero but not injective, so has a kernel and cokernel isomorphic to $\mathbb{Z}$, which shows that $H_1(S^1) \cong H_0(S^1) \cong \mathbb{Z}$. For $n \geq 1$, one shows that the intersection $A_n \cap B_n$ is homotopic to S^{n-1}. Thus using the Mayer-Vietoris repeatedly proves the theorem. □

We illustrate some of the concepts and methods with an example describing the use of homology to prove Brouwer's fixpoint theorem.

Theorem 3.3.8 (Brouwer's fixpoint theorem). *Let m be a positive integer. Any continuous map $f : D^m \to D^m$ has a fixpoint (that is, there is $x \in D^m$ such that $f(x) = x$).*

Proof. For $m = 1$ this is an easy application of the intermediate value theorem. Suppose that $m \geq 2$. Suppose the theorem is not true; that is, there is a continuous map $f : D^m \to D^m$ such that $f(x) \neq x$ for any $x \in D^m$. Since $f(x)$ is a point different from x in the disc D^m, there is a unique line starting from $f(x)$ and passing through x. This line intersects the boundary S^{m-1} of D^m in a point which we denote by $g(x)$. One verifies that the map $x \mapsto g(x)$ is a continuous map $g : D^m \to S^{m-1}$. If x is on that boundary S^{m-1}, then clearly $g(x) = x$. That is, $g : D^m \to S^{m-1}$ restricts to the identity map on S^{m-1}. In other words, if we denote by $a : S^{m-1} \to D^m$ the inclusion map, then $g \circ a = \mathrm{Id}_{S^{m-1}}$. Applying H_n for any $n \geq 0$ and using the functoriality properties of H_n yields $H_n(g) \circ H_n(a) = \mathrm{Id}_{H_n(S^{m-1})}$. For $n = m-1$ this yields a contradiction: the right side is the identity map on the nonzero abelian group $H_{m-1}(S^{m-1})$, but the map $H_{m-1}(a)$ is zero, because $H_{m-1}(D^m)$ is zero, and so $H_n(g) \circ H_n(a)$ is zero as well. □

There are other statements that can be proved in a similar way: for $n > 0$, the sphere S^{n-1} is not contractible, two spheres S^{n-1}, S^{m-1} for positive integers m, n are isomorphic if and only if $n = m$, and $\mathbb{R}^n$, R^m are isomorphic as topological spaces if and only if $n = m$.

Chapter 4

Triangulated categories

4.1 Triangulated categories

A triangulated category is an additive category with an additional structure of *exact triangles*, which should be thought of as a replacement for short exact sequences. This concept, which we will introduce in the present section, has been developed independently by J.L. Verdier, and, in a topological context, by D. Puppe. We will show in the next section that homotopy categories of chain complexes are triangulated.

Given an additive category $\mathcal{C}$ and an additive functor $\Sigma : \mathcal{C} \to \mathcal{C}$ on $\mathcal{C}$, we call a *triangle in* $\mathcal{C}$ a sequence of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

where X, Y, Z are objects in $\mathcal{C}$ and f, g, h are morphisms in $\mathcal{C}$. The triangles in $\mathcal{C}$ form the objects of a category: a *morphism of triangles* is a triple (u, v, w) of morphisms in $\mathcal{C}$ making the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma(u)} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma(X') \end{array}$$

commutative, where the two rows are triangles in $\mathcal{C}$.

Definition 4.1.1. A *triangulated category* is a triple $(\mathcal{C}, \Sigma, \mathcal{T})$ consisting of an additive category $\mathcal{C}$, a covariant additive self equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$ and a class $\mathcal{T}$ of triangles in $\mathcal{C}$ - called *exact* or sometimes also *distinguished triangles* in $\mathcal{C}$ - fulfilling the axioms T1, T2, T3, T4 below.

T1: For any object X in $\mathcal{C}$, the triangle $0 \longrightarrow X \xrightarrow{\mathrm{Id}_X} X \longrightarrow 0$ is exact (i.e., belongs to the class $\mathcal{T}$), for any morphism $f : X \to Y$ in $\mathcal{C}$ there is an exact triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

for some object Z in $\mathcal{C}$ and some morphisms g, h, any triangle in $\mathcal{C}$ which is isomorphic to an exact triangle is itself exact (i.e., the class $\mathcal{T}$ is closed under isomorphisms).

T2: Any commutative diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & & & \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma(X') \end{array}$$

whose rows are exact triangles, can be completed to a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma(u)} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma(X') \end{array}$$

for some morphism w.

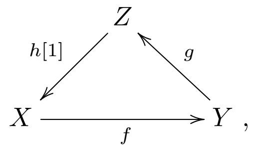
T3: If the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ in $\mathcal{C}$ is exact, so is the triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma f} \Sigma(Y)$$

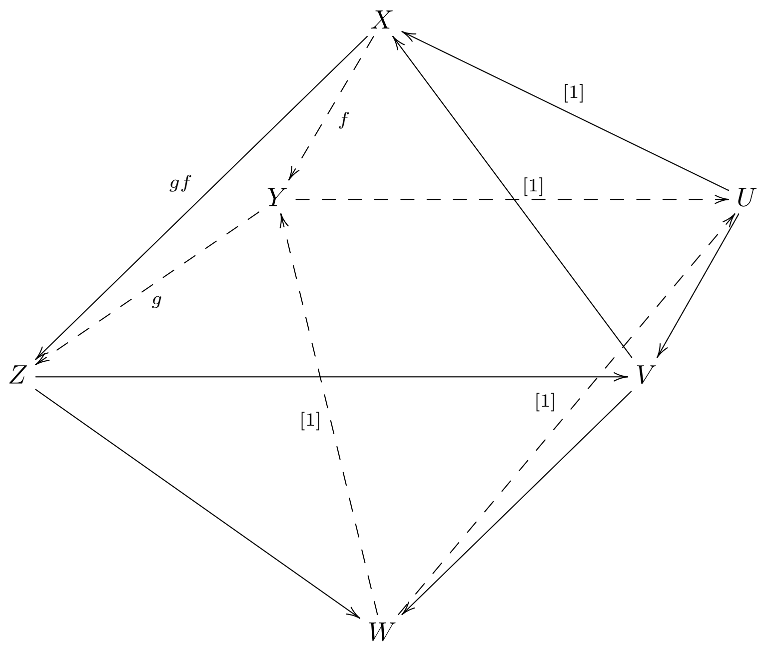
T4: Given any sequence of two composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$ there is a commutative diagram in $\mathcal{C}$ whose first two rows and middle two columns are exact triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{k} & U & \longrightarrow & \Sigma(X) \\ \| & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} & & \| \\ X & \xrightarrow[gf]{} & Z & \longrightarrow & V & \longrightarrow & \Sigma(X) \\ & & \downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \Sigma(f)} \\ & & W & = & W & \xrightarrow[w]{} & \Sigma Y \\ & & \downarrow{\scriptstyle w} & & \downarrow & & \\ & & \Sigma(Y) & \xrightarrow[\Sigma(k)]{} & \Sigma(U) & & \end{array}$$

Remark 4.1.2. The axiom T4 describes in which way the three triangles over f, g, $g \circ f$ are connected. This axiom is called the *octahedral axiom* for the following reason: if we rewrite a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ in the form



where [1] means that h "is of degree 1", then the diagram in T4 takes the following form:



Proposition 4.1.3. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ *be an exact triangle in a triangulated category* $\mathcal{C}$*, and let* U *be an object in* $\mathcal{C}$*.*

(i) We have $g \circ f = 0$ *and* $h \circ g = 0$*.*

(ii) Given any morphism $j : Y \to U$ *there is a morphism* $i : Z \to U$ *satisfying* $i \circ g = j$ *if and only if* $j \circ f = 0$*.*

(iii) Given any morphism $q : U \longrightarrow Y$ *there is a morphism* $p : U \to X$ *satisfying* $f \circ p = q$ *if and only if* $g \circ q = 0$*.*

Proof. By T1 the triangle $0 \longrightarrow X = X \longrightarrow 0$ is exact, and hence, by T3, the triangle $X = X \longrightarrow 0 \longrightarrow \Sigma(X)$ is exact. Applying T2 yields the existence of a commutative diagram

$$\begin{array}{ccccccc}
X & = & X & \longrightarrow & 0 & \longrightarrow & \Sigma(X) \\
\| & & \downarrow{\scriptstyle f} & & \downarrow & & \| \\
X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & \Sigma(X)
\end{array}$$

which shows that $g \circ f = 0$. The same argument, after turning the triangle by means of T3, shows that $h \circ g = 0$, whence (i). If $j \circ f = 0$, then it follows from T1 and T2 that we have a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\
\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow \\
0 & \longrightarrow & U & = & U & \longrightarrow & 0
\end{array}$$

for some morphism i, which means that $i \circ g = j$. Conversely, if there is a morphism $i : Z \to U$ such that $i \circ g = j$, then $j \circ f = i \circ g \circ f = 0$, since $g \circ f = 0$ by (i). This proves (ii). The last statement is proved by applying a dual argument to the exact triangle $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma(f)} \Sigma(Y)$. □

Lemma 4.1.4. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category, and let*

$$\begin{array}{ccccccc}
 & & Y & \xrightarrow{g} & Z & & \\
 & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle 0} & & \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X') \\
{\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle v'} & & & & \\
X'' & \xrightarrow[f'']{} & Y'' & & & &
\end{array}$$

be a commutative diagram in $\mathcal{C}$ whose middle row is an exact triangle. We have $v' \circ v = 0$.

Proof. Since $g' \circ v = 0$ there is, by 4.1.3, a morphism $p : Y \to X'$ such that $f' \circ p = v$. Similiarly, since $v' \circ f' = 0$ there is a morphism $i : Z' \to Y''$ such that $v' = i \circ g'$. Together we obtain $v' \circ v = i \circ g' \circ f \circ p = 0$, since $g' \circ f' = 0$ by 4.1.3. □

Proposition 4.1.5. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category and let*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\
{\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \Sigma(u)} \\
X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma(X')
\end{array}$$

be a commutative diagram in $\mathcal{C}$ whose rows are exact triangles. If u, v are isomorphisms, so is w.

Proof. By applying T2 to u^{-1} and v^{-1} we may assume that $X = X'$, $Y = Y'$, $Z = Z'$, $f = f'$, $g = g'$, $h = h'$, $u = \mathrm{Id}_X$, and $v = \mathrm{Id}_Y$. Thus we are down to considering the endomorphism $(\mathrm{Id}_X, \mathrm{Id}_Y, w)$ of the exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$, and we have to show that w is an automorphism. Clearly $(\mathrm{Id}_X, \mathrm{Id}_Y, \mathrm{Id}_Z)$ is an endomorphism of this triangle, too, thus taking the difference of the two endomorphisms yields an endomorphism $(0, 0, \mathrm{Id}_Z - w)$. Using T3 and 4.1.4 shows that $(\mathrm{Id}_Z - w)^2 = 0$, or equivalently, $\mathrm{Id}_Z = w \circ (2\mathrm{Id}_Z - w)$, which implies that w is invertible with inverse $2\mathrm{Id}_Z - w$. □

Corollary 4.1.6. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category. If the triangle*

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma f} \Sigma(Y)$$

is exact in $\mathcal{C}$, so is the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

Proof. By T1, there is an exact triangle in $\mathcal{C}$ of the form $X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma(X)$. We turn this triangle three times, and the first triangle in the statement twice; this yields two exact triangles in $\mathcal{C}$

$$\Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g'} \Sigma Z' \xrightarrow{-\Sigma h'} \Sigma^2(X)$$

$$\Sigma(X) \xrightarrow{-\Sigma f} \Sigma(Y) \xrightarrow{-\Sigma g} \Sigma Z \xrightarrow{-\Sigma h} \Sigma^2(X)$$

and by Proposition 4.1.5, these two triangles are isomorphic. Since Σ is an equivalence, it follows that the triangles

$$X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma(X)$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z' \xrightarrow{h} \Sigma(X)$$

are isomorphic, and as the first one is exact, so is the second by T1. □

Corollary 4.1.7. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category, and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

be an exact triangle in $\mathcal{C}$.

(i) f is an isomorphism if and only if $Z = 0$.

(ii) g is an isomorphism if and only if $X = 0$.

(iii) h is an isomorphism if and only if $Y = 0$.

Proof. If f is an isomorphism, then, by 4.1.5 and T1, the given exact triangle is isomorphic to the exact triangle $X = X \longrightarrow 0 \longrightarrow \Sigma(X)$, thus $Z = 0$. Conversely, if $Z = 0$, turning the triangle by T3 shows that $\Sigma(f)$ is an isomorphism, and hence f is so, too, as Σ is an equivalence. This shows (i), and the other statements follow from (i) with T3 and the fact, that Σ is an equivalence. □

Corollary 4.1.8. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

be an exact triangle in $\mathcal{C}$. For any object U in $\mathcal{C}$, the functors $\mathrm{Hom}_{\mathcal{C}}(U, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, U)$ induce long exact sequences of abelian groups

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^n(X)) \to \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^n(Y)) \to \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^n(Z)) \to \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^{n+1}(X)) \to \cdots$$

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(\Sigma^{n+1}(X), U) \to \mathrm{Hom}_{\mathcal{C}}(\Sigma^n(Z), U) \to \mathrm{Hom}_{\mathcal{C}}(\Sigma^n(Y), U) \to \mathrm{Hom}_{\mathcal{C}}(\Sigma^n(X), U) \to \cdots$$

Proof. By 4.1.3, the sequence $\mathrm{Hom}_{\mathcal{C}}(U, X) \to \mathrm{Hom}_{\mathcal{C}}(U, Y) \to \mathrm{Hom}_{\mathcal{C}}(U, Z)$ is exact. Turning the triangle by means of T3 and its converse 4.1.6 yields the first of the two long exact sequences. An analogous argument shows the exactness of the second sequence. □

Proposition 4.1.9. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category and let $f : X \to Y$ be a morphism in $\mathcal{C}$.*

(i) f is an epimorphism if and only if f has a right inverse.

(ii) f is a monomorphism if and only if f has a left inverse.

(iii) f is an isomorphism if and only if f is both an epimorphism and a monomorphism.

Proof. If f has a right inverse, f is trivially an epimorphism. Conversely, suppose that f is an epimorphism. Consider an exact triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

Since $g \circ f = 0$ and since f is an epimorphism, we have $g = 0$. Thus $g \circ \mathrm{Id}_Y = 0$. Thus there is $r : Y \to X$ such that $f \circ r = \mathrm{Id}_Y$, which shows that r is a right inverse of f. This shows (i), and a dual argument shows statement (ii). If f is both an epimorphism and a monomorphism, it has a right inverse r and a left inverse l, and then $l = l \circ f \circ r = r$, which shows that f is an isomorphism. The converse in (iii) is trivial. □

This shows that epimorphisms and monomorphisms in a triangulated category are all split. As a consequence, a triangulated category is abelian if and only if it is semisimple. There are, however, nontrivial examples of proper abelian subcategories of triangulated categories arising in the context of central p-extensions of finite groups. See [9] for a systematic treatment of triangulated categories.

4.2 Homotopy categories are triangulated

If $\mathcal{C}$ is an additive category, then the chain homotopy category $K(\mathcal{C})$ together with the shift functor [1] is triangulated. To see this we need to define a suitable class of exact triangles.

Definition 4.2.1. Let $\mathcal{C}$ be an additive category and let $f : (X, \delta) \to (Y, \epsilon)$ be a chain map of complexes over $\mathcal{C}$. The *mapping cone of* f is the complex $C(f)$ over $\mathcal{C}$ given by $C(f)_n = X_{n-1} \oplus Y_n$ with differential Δ

$$\Delta_n = \begin{pmatrix} -\delta_{n-1} & 0 \\ f_{n-1} & \epsilon_n \end{pmatrix} : X_{n-1} \oplus Y_n \to X_{n-2} \oplus Y_{n-1}$$

for any integer n. The mapping cone comes along with canonical chain maps $i(f) : Y \to C(f)$ given by the canonical monomorphims $Y_n \hookrightarrow X_{n-1} \oplus Y_n$ and $p(f) : C(f) \to X[1]$ given by the canonical projections $X_{n-1} \oplus Y_n \twoheadrightarrow X_{n-1} = X[1]_n$ for any integer n.

One checks that $\Delta \circ \Delta = 0$. The *cone* $C(X)$ *of a chain complex* X is the mapping cone of the identity chain map Id_X.

Associated with any chain map $f : X \to Y$ we have a triangle in $(\mathrm{Ch}(\mathcal{C}), [1])$ given by the "mapping cone sequence"

$$X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{p(f)} X[1]$$

and we denote by $\mathcal{T}$ the class of all triangles in $K(\mathcal{C})$ isomorphic to the image of a triangle in $\mathrm{Ch}(\mathcal{C})$ of this form. We are going to show that this yields a structure of a triangulated category for $K(\mathcal{C})$.

Theorem 4.2.2. *Let $\mathcal{C}$ be an additive category. Then $K(\mathcal{C})$, endowed with the shift functor [1] and the class $\mathcal{T}$ of triangles induced by mapping cone sequences, is a triangulated category. Moreover, the categories $K^+(\mathcal{C})$, $K^-(\mathcal{C})$, $K^b(\mathcal{C})$ are full triangulated subcategories in $K(\mathcal{C})$.*

Axiom T1 holds trivially: by definition, any chain map is part of a mapping cone sequence, and the mapping cone of the zero map $0 \to X$ is just X again, so the mapping cone sequence degenerates to $0 \to X \to X \to 0$. The following Proposition shows, that T2 holds:

Proposition 4.2.3. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

be a commutative diagram of chain complexes over an additive category $\mathcal{C}$. Then there is a chain map $w : C(f) \to C(f')$ making the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{p(f)} & X[1] \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[i(f')]{} & C(f') & \xrightarrow[i(f')]{} & X'[1] \end{array}$$

homotopy commutative.

Proof. Since $v \circ f \sim f' \circ u$, there is a homotopy $h : X \to Y'$ such that $v \circ f - f' \circ u = \epsilon' \circ h + h \circ \delta$, where δ and ϵ' are the differentials of X and Y', respectively. Set

$$w_n = \begin{pmatrix} u_{n-1} & 0 \\ h_{n-1} & v_n \end{pmatrix} : X_{n-1} \oplus Y_n \to X'_{n-1} \oplus Y'_n$$

for any $n \in \mathbb{Z}$. A straightforward verification shows that $w = (w_n)_{n \in \mathbb{Z}}$ is a chain map from $C(f)$ to $C(f')$ which makes the middle and right square in the above diagram commutative. □

The next Proposition describes in which way the mapping cones $C(f)$, $C(i(f))$, $C(p(f))$ are connected, implying in particular, that the axiom T3 holds for the class $\mathcal{T}$ in $K(\mathcal{C})$:

Proposition 4.2.4. *Let $f : X \to Y$ be a chain map of chain complexes over an additive category $\mathcal{C}$. Denote by $q(f) : C(i(f)) \to X[1]$ the graded map given by the canonical projections $q(f)_n = (0, \mathrm{Id}_{X_{n-1}}, 0) : Y_{n-1} \oplus (X_{n-1} \oplus Y_n) \to X_{n-1}$, for any $n \in \mathbb{Z}$. Denote by $s(f) : C(p(f)) \to Y[1]$ the graded map given by $s(f)_n = (0, \mathrm{Id}_{Y_{n-1}}, f_{n-1}) : X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \to Y_{n-1}$, for any $n \in \mathbb{Z}$. Then $q(f)$ and $s(f)$ are homotopy equivalences making the following diagram of chain complexes homotopy commutative:*

$$
\begin{array}{ccccccccccc}
 & & Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{i(i(f))} & C(i(f)) & \xrightarrow{p(i(f))} & Y[1] & & \\
 & & \| & & \| & & \downarrow{\scriptstyle q(f)} & & \| & & \\
X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{p(f)} & X[1] & \xrightarrow{-f[1]} & Y[1] & \xrightarrow{-i(f)[1]} & C(f)[1] \\
 & & & & \| & & \| & & \uparrow{\scriptstyle -s(f)} & & \| \\
 & & & & C(f) & \xrightarrow[p(f)]{} & X[1] & \xrightarrow[i(p(f))]{} & C(p(f)) & \xrightarrow[p(p(f))]{} & C(f)[1]
\end{array}
$$

Proof. The verification, that both $q(f)$, $s(f)$ are chain maps, is straightforward. We construct homotopy inverses $r(f)$, $t(f)$ of $q(f)$, $s(f)$, respectively, as follows. Set

$$r(f)_n = \begin{pmatrix} -f_{n-1} \\ \mathrm{Id}_{X_{n-1}} \\ 0 \end{pmatrix} : X_{n-1} \to Y_{n-1} \oplus X_{n-1} \oplus Y_n$$

for any integer n. Then $r(f)$ is a chain map satisfying $q(f) \circ r(f) = \mathrm{Id}_{X[1]}$. In order to show that $r(f) \circ q(f) \sim \mathrm{Id}_{C(i(f))}$, we define the homotopy h on $C(i(f))$ by

$$h_n = \begin{pmatrix} 0 & 0 & \mathrm{Id}_{Y_n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : Y_{n-1} \oplus X_{n-1} \oplus Y_n \to Y_n \oplus X_n \oplus Y_{n+1}$$

for any integer n. If Δ denotes the differential of $C(i(f))$, we have $\Delta \circ h + h \circ \Delta = \mathrm{Id}_{C(i(f))} - r(f) \circ q(f)$. This shows also the homotopy commutativity of the upper part of the diagram, since $q(f) \circ i(i(f)) = p(f)$ and $p(i(f)) \circ r(f) = -f[1]$. We proceed similarly for $t(f)$. Set

$$t(f)_n = \begin{pmatrix} 0 \\ \mathrm{Id}_{Y_{n-1}} \\ 0 \end{pmatrix} : Y_{n-1} \to X_{n-2} \oplus Y_{n-1} \oplus X_{n-1}$$

for any integer n. Clearly $t(f)$ is a chain map satisfying $s(f) \circ t(f) = \mathrm{Id}_{Y[1]}$. In order to show that $t(f) \circ s(f) \sim \mathrm{Id}_{C(p(f))}$, we define the homotopy k on $C(p(f))$ by

$$k_n = \begin{pmatrix} 0 & 0 & \mathrm{Id}_{X_{n-1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \to X_{n-1} \oplus Y_n \oplus X_n$$

for any integer n. If Π denotes the differential of $C(p(f))$, we have $\Pi \circ k + k \circ \Pi = \mathrm{Id}_{C(p(f))} - t(f) \circ s(f)$. This shows also the homotopy commutativity of the lower part of the diagram, since $s(f) \circ p(f) = f[1]$ and $p(p(f)) \circ t(f) = i(f)[1]$. □

It remains to show, that the octahedral axiom T4 holds:

Proposition 4.2.5. *Given two composable chain maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *of complexes over an additive category* $\mathcal{C}$*, there are chain maps* u*,* v *making the following diagram of chain complexes commutative*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{p(f)} & X[1] \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \| \\
X & \xrightarrow{gf} & Z & \xrightarrow{i(gf)} & C(gf) & \xrightarrow{p(gf)} & X[1] \\
 & & \downarrow{\scriptstyle i(g)} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f[1]} \\
 & & C(g) & = & C(g) & \xrightarrow{p(g)} & Y[1] \\
 & & \downarrow{\scriptstyle p(g)} & & \downarrow{\scriptstyle w} & & \\
 & & Y[1] & \xrightarrow{i(f)[1]} & C(f)[1] & &
\end{array}$$

and there is a homotpy equivalence $t(u) : C(g) \to C(u)$ *such that the diagram*

$$\begin{array}{ccccccc}
C(f) & \xrightarrow{u} & C(gf) & \xrightarrow{v} & C(g) & \xrightarrow{w} & C(f)[1] \\
\| & & \| & & \downarrow{\scriptstyle t(u)} & & \| \\
C(f) & \xrightarrow{u} & C(gf) & \xrightarrow{i(u)} & C(u) & \xrightarrow{p(u)} & C(f)[1]
\end{array}$$

is homotopy commutative.

Proof. For any $n \in \mathbb{Z}$ set

$$u_n = \begin{pmatrix} \mathrm{Id}_{X_{n-1}} & 0 \\ 0 & g_n \end{pmatrix} : X_{n-1} \oplus Y_n \to X_{n-1} \oplus Z_n \quad ,$$

$$v_n = \begin{pmatrix} f_{n-1} & 0 \\ 0 & \mathrm{Id}_{Z_n} \end{pmatrix} : X_{n-1} \oplus Z_n \to Y_{n-1} \oplus Z_n \quad ,$$

$$w_n = \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_{Y_{n-1}} & 0 \end{pmatrix} : Y_{n-1} \oplus Z_n \to X_{n-2} \oplus Y_{n-1} \quad .$$

A straightforward verification shows that $u = (u_n)_{n\in\mathbb{Z}}$, $v = (v_n)_{n\in\mathbb{Z}}$ and $w = (w_n)_{n\in\mathbb{Z}}$ are chain maps which make the first diagram in the statement commutative. For $t(u)$ we take the morphism given by the obvious split monomorphisms

$$Y_{n-1} \oplus Z_n \longrightarrow (X_{n-2} \oplus Y_{n-1}) \oplus (X_{n-1} \oplus Z_n)$$

and we define a morphism $s(u) : C(u) \to C(g)$ given by the projections

$$(X_{n-2} \oplus Y_{n-1}) \oplus (X_{n-1} \oplus Z_n) \longrightarrow Y_{n-1} \oplus Z_n$$

for any $n \in \mathbb{Z}$. Then $s(u) \circ t(u) = \mathrm{Id}_{C(g)}$, and it remains to show that $t(u) \circ s(u) \sim \mathrm{Id}_{C(u)}$. For this we consider on $C(u)$ the homotopy h given, for any $n \in \mathbb{Z}$, by the map

$$h_n : (X_{n-2} \oplus Y_{n-1}) \oplus (X_{n-1} \oplus Z_n) \to (X_{n-1} \oplus Y_n) \oplus (X_n \oplus Z_{n+1}) \quad ,$$

where h_n is zero on the summands X_{n-2}, Y_{n-1}, Z_n, and h_n maps X_{n-1} identically to its canonical image in $C(u)_{n+1}$. Then in the second diagram in the statement, the left and middle square are commutative. Clearly $p(v) \circ s(u) = p(u)$; thus the right square is homotopy commutative, as $s(u)$ is a homotopy inverse to $t(u)$. □

This completes the proof of Theorem 4.2.2. We note some immediate consequences.

Corollary 4.2.6. *Let $\mathcal{C}$ be an additive category, let $f : X \to Y$ be a chain map of complexes over $\mathcal{C}$, and consider the mapping cone sequence* $X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{p(f)} X[1]$.

(i) We have $i(f) \circ f \sim 0$.

(ii) f is a homotopy equivalence if and only if $C(f) \simeq 0$.

(iii) $i(f)$ is a homotopy equivalence if and only if $X \simeq 0$.

(iv) $p(f)$ is a homotopy equivalence if and only if $Y \simeq 0$.

(v) If two of X, Y, $C(f)$ are homotopic to zero, so is the third.

Proof. Statement (i) follows from 4.1.3 (i), but one can see this also directly: the canonical monomorphisms $X_n \hookrightarrow X_n \oplus Y_{n+1}$ define a homotopy $h : X \longrightarrow C(f)$ through which $i(f) \circ f$ becomes homotopic to the zero map. The statements (ii), (iii), (iv) are all particular cases of 4.1.7. Finally, (v) follows from (ii) and (iii). □

Corollary 4.2.7. *Let $\mathcal{C}$ be an additive category, $f : X \to Y$ a chain map of complexes over $\mathcal{C}$, and let U be a complex over $\mathcal{C}$.*

(i) The covariant functor $\mathrm{Hom}_{K(\mathcal{C})}(U, -)$ induces a long exact sequence

$$\cdots \to \mathrm{Hom}_{K(\mathcal{C})}(U, X[n]) \to \mathrm{Hom}_{K(\mathcal{C})}(U, Y[n]) \to \mathrm{Hom}_{K(\mathcal{C})}(U, C(f)[n]) \to \mathrm{Hom}_{K(\mathcal{C})}(U, X[n+1]) \to \cdots$$

(ii) The contravariant functor $\mathrm{Hom}_{K(\mathcal{C})}(-, U)$ induces a long exact sequence

$$\cdots \to \mathrm{Hom}_{K(\mathcal{C})}(X[n+1], U) \to \mathrm{Hom}_{K(\mathcal{C})}(C(f)[n], U) \to \mathrm{Hom}_{K(\mathcal{C})}(Y[n], U) \to \mathrm{Hom}_{K(\mathcal{C})}(X[n], U) \to \cdots$$

Proof. This is a particular case of 4.1.8. □

Corollary 4.2.8. *Let A be an algebra over a commutative ring k and let $f : X \to Y$ be a chain map of complexes of A-modules. Taking homology induces a long exact sequence of A-modules*

$$\cdots \to H_n(X) \to H_n(Y) \to H_n(C(f)) \to H_{n-1}(X) \to \cdots$$

Proof. By 1.3.11 we have $H_n(X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A[n], X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A, X[-n])$, thus the statement follows from the first of the two long exact sequences in the previous corollary applied to $U = A$. □

Corollary 4.2.9. *Let A be an algebra over a commutative ring k and let $f : X \to Y$ be a chain map of complexes of A-modules. The following are equivalent.*
(i) f is a quasi-isomorphism.
(ii) $C(f)$ is acyclic.
(iii) For any bounded below complex P of projective A-modules, the map f induces an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Y)$.
(iv) For any bounded above complex I of injective A-modules, the map f induces an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(Y, I) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, I)$.

Proof. The equivalence of (i), (ii) follows from the long exact homology sequence in the previous corollary, and the equivalence with (iii), (iv) follows then from the long exact sequences in 4.2.7, together with the characterisation 1.3.10 of acyclic complexes, using the fact that $\mathrm{Mod}(A)$ has enough projective and injective objects. □

We have two ways of producing long exact sequences: via mapping cone sequences and via short exact sequences of complexes. Both approaches are equivalent in the sense that we can view mapping cone sequences as being induced by short exact sequences of complexes in the same way we defined triangles in a stable module category using short exact sequences of modules.

Theorem 4.2.10. *Let A be an algebra over a commutative ring k and let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of chain complexes A-modules. The maps $s_n = (0, g_n) : X_{n-1} \oplus Y_n \to Z_n$ induce a quasi-isomorphism $s : C(f) \to Z$ making the diagram of chain complexes of A-modules

$$\begin{array}{ccccccccc}
 & & X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C(f) & & \\
 & & \| & & \| & & \downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0
\end{array}$$

commutative, and we have an isomorphism of long exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_n(X) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(C(f)) & \longrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-1}(Y) & \longrightarrow & \cdots \\
 & & \| & & \| & & \downarrow{\scriptstyle H_n(s)} & & \| & & \| & & \\
\cdots & \longrightarrow & H_n(X) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Z) & \longrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-1}(Y) & \longrightarrow & \cdots
\end{array}$$

where the first row is from 4.2.8 and the second row from 1.2.8. Moreover, if the first exact sequence of chain complexes is degreewise split, then s is a homotopy equivalence.

Proof. The commutativity of the diagram is a straightforward verification. One can verify either directly that s is a quasi-isomorphism (which, effectively, would yield another proof of Theorem 1.2.8), or use the 5-Lemma. Denote by δ, ϵ, ζ, Δ the differentials of X, Y, Z, $C(f)$, respectively. Suppose now that the first exact sequence of chain complexes in the statement is degree wise split; that is, there are graded morphisms $u : Y \to X$ and $v : Z \to Y$ satisfying $\mathrm{Id}_Y = f \circ u + v \circ g$. Note that then $f = f \circ u \circ f$, hence $u \circ f = \mathrm{Id}_X$ as f is a monomorphism; in particular, u is a retraction for f. Similarly, $g \circ v = \mathrm{Id}_Z$; in particular, v is a section for g. The morphism $v \circ \zeta - \epsilon \circ v : Z \to Y$ is graded of degree -1, and it this a chain map from Z to $Y[1]$, since

$$(-\epsilon) \circ (v \circ \zeta - \epsilon \circ v) = -\epsilon \circ v \circ \zeta = (v \circ \zeta - \epsilon \circ v) \circ \zeta$$

This chain map satisfies $g \circ (v \circ \zeta - \epsilon \circ v) = g \circ v \circ \zeta - \zeta \circ g \circ v = 0$, as $g \circ v = \mathrm{Id}_Z$. Whence this map factors through f. Let $r : Z \to X$ be the graded morphism of degree -1 such that $f \circ r = v \circ \zeta - \epsilon \circ v$. Since the right side is a chain map and f is a monomorphism, r itself can be viewed as a chain map from $Z[-1]$ to X. Consider the associated triangle

$$Z[-1] \xrightarrow{r} X \xrightarrow{i(r)} C(r) \xrightarrow{p(r)} Z$$

A straightforward verification shows that $C(r) \cong Y$ via the inverse chain maps given by the morphisms $(v_n, f_n) : Z_n \oplus X_n \to Y_n$ and $\begin{pmatrix} g_n \\ u_n \end{pmatrix} : Y_n \to Z_n \oplus X_n$ for any integer n. Thus $C(i(r)) \cong C(f)$. By 4.2.4, we have also a homotopy equivalence $q(r) : C(i(r)) \to Z$. Together, we obtain a homotopy equivalence $C(f) \simeq Z$, and this is easily seen to be the chain map s as defined. □

Corollary 4.2.11. *With the notation and hypotheses of 4.2.10, if $Y \simeq 0$ then $Z \simeq X[1]$.*

Chapter 5

Spectral sequences

Spectral sequences, introduced by Jean Leray, are a sophisticated tool to calculate the (co-)homology of (co-)chain complexes in terms of a filtration by subcomplexes. If X is a subcomplex of a cochain complex Y, then the cohomology of Y is related to that of X and Y/X via the long exact homology sequence 1.2.8. More generally, if we have a filtration of a cochain complex X by subcomplexes F^iX such that F^iX is a subcomplex of $F^{i-1}X$, then the cohomology of X can be approximated in terms of the cohomology of the quotiens F^iX/F^jX, with $j > i$. A spectral sequence organises the data coming from such a filtration in a way which leads to calculating a filtration of the cohomology of X in terms of the given filtration of X itself. We have, of course, as always the dual version for the homology of chain complexes. We describe here cohomology spectral sequences and leave the translation of this material to homology sequences as an exercise. We fix an algebra A over a commutative ring k and describe spectral sequences of cochain complexes of A-modules; the adaptation to arbitrary abelian categories is straightforward.

5.1 Spectral sequences: definition

Definition 5.1.1. A *cohomology spectral sequence starting at the page* E_a for some integer a is a triple graded family of A-modules $E = (E_r^{p,q})$, where $r \geq a$ and p, $q \in \mathbb{Z}$, together with A-homomorphisms

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all r, p, q as before, together with an isomorphism between

$$E_{r+1}^{p,q} \cong \ker(d_r^{p,q})/\mathrm{Im}(d_r^{p-r,q+r-1})$$

For a fixed r, the bigraded A-module $E_r^{*,*} = (E_r^{p,q})_{p,q\in\mathbb{Z}}$ together with the differentials $d_r^{p,q}$ is called the *E_r-page of the spectral sequence.* The number $p + q$ is called the *total degree* of the A-module $E_r^{p,q}$ in the spectral sequence. The cohomology spectral sequences starting at a fixed page E_a are the objects of a category in which a morphism of spectral sequences $(E_r^{p,q}) \to (F_r^{p,q})$ is a triple graded A-homomorphism $(f_r^{p,q} : E_r^{p,q} \to F_r^{p,q})$ which commute with the differentials and the isomorphisms $E_{r+1}^{p,q} \cong \ker(d_r^{p,q})/\mathrm{Im}(d_r^{p-r,q+r-1})$.

The E_r-page of a spectral sequence is a family of complexes with differentials of bidegree $(r, -r+1)$, and the passage from the E_r-page to the E_{r+1}-page is made by taking the cohomology of the involved complexes. Note that d^{p,q_r} goes from $E_r^{p,q}$ with total degree $p+q$ to $E_r^{p+r,q-r+1}$ with total degree $p+q+1$. For E a spectral sequence starting at the E_0 page, the pages E_0, E_1, E_2 can be visualised as lattices of A-modules together with their differentials as follows. The bidegree of the differential of E_0 is $(0,1)$.

$$
\begin{array}{ccc}
E_0^{0,2} & E_0^{1,2} & E_0^{2,2} \\
\uparrow & \uparrow & \uparrow \\
E_0^{0,1} & E_0^{1,1} & E_0^{2,1} \\
\uparrow & \uparrow & \uparrow \\
E_0^{0,0} & E_0^{1,0} & E_0^{2,0}
\end{array}
$$

Any two vertical maps compose to zero, and the resulting cohomology is yields the terms of the next page E_1, whose differential has bidegree $(1,0)$.

$$
\begin{array}{ccccc}
E_1^{0,2} & \longrightarrow & E_1^{1,2} & \longrightarrow & E_1^{2,2} \\
E_1^{0,1} & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} \\
E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0}
\end{array}
$$

Again, any two consecutive horizontal maps compose to zero, and the resulting cohomology yields the terms of the next page E_2, whose differential has bidegree $(2,-1)$.

$$
\begin{array}{ccccc}
E_2^{0,2} & E_2^{1,2} & E_2^{2,2} & E_2^{3,2} & E_2^{4,2} \\
 & \searrow\ \ \searrow & \searrow & & \\
E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & E_2^{4,1} \\
 & \searrow\ \ \searrow & \searrow & & \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & E_2^{4,0}
\end{array}
$$

Observation: if for some fixed p, q we have $E_r^{p,q} = 0$, then also $E_s^{p,q}$ for all $s \geq r$. That is, any zero entry in a given page remains zero in all subsequent pages. More generally, the module $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$. If for any $(p,q) \in \mathbb{Z}^2$ the process of taking subquotients stabilises for some sufficiently large r, then we say that the spectral sequence converges. More precisely:

Definition 5.1.2. A cohomology spectral sequence $(E_r^{p,q})$ with start page E_a *converges* if for any $(p,q) \in \mathbb{Z}^2$ there exists $r \geq a$ such that $E_s^{p,q} = E_r^{p,q}$ for all $s \geq r$. In that case, we write $E_r^{p,q} = E_\infty^{p,q}$.

For a spectral sequence to be useful, convergence is a key property, and so quite some effort goes into developing sufficient criteria for a spectral sequence to converge. We consider the following two cases.

Definition 5.1.3. A spectral sequence $(E_r^{p,q})$ starting at the page E_a is called *bounded* if for any $n \in \mathbb{Z}$ there are only finitely many $(p,q) \in \mathbb{Z}$ such that $p+q=n$ and $E_a^{p,q} \neq 0$.

As remarked earlier, this implies that for all $r \geq a$, the page E_r satisfies the same boundedness property as the page E_a.

Definition 5.1.4. A spectral sequence $(E_r^{p,q})$ starting at the page E_a is called a *first quadrant spectral sequence* if $E_a^{p,q} = 0$ for all $(p,q) \in \mathbb{Z}$ such that $p < 0$ or $q < 0$.

The term 'first quadrant spectral sequence' is self-explanatory: a first quadrant spectral sequence $(E_r^{p,q})$ has in any page nonzero terms $E_r^{p,q}$ only if $p \geq 0$ and $q \geq 0$, that is, only if (p,q) belongs to the first quadrant.

Exercise 5.1.5. Let $E = (E_r^{p,q})$ be a first quadrant spectral sequence starting at the page E_0. Show that for any integer $n \geq 0$ and any $(p,q) \in \mathbb{Z}^2$ such that $p+q=n$ we have $E_\infty^{p,q} = E_{n+2}^{p,q}$.

Proposition 5.1.6.

(i) A first quadrant spectral sequence is bounded.

(ii) A bounded spectral sequence converges.

Proof. For any integer n there are at most finitely many pairs $(p,q) \in \mathbb{Z}^2$ such that $p \geq 0$, $q \geq 0$, and $p+q=n$. Thus a first quadrant spectral sequence is bounded, whence (i). For (ii), let $(E_r^{p,q})$ be a bounded spectral sequence. Let $n \in \mathbb{Z}$ and $(p,q) \in \mathbb{Z}^2$ such that $p+q=n$. Consider the sequence of the two differentials in the E_r-page starting and ending at $E_r^{p,q}$,

$$E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

Regardless of r, the total degree of the left term $E_r^{p-r,q+r-1}$ is $n-1$, and the total degree of the right term $E_r^{p+r,q-r+1}$ is $n+1$. Since the spectral sequence is bounded, there are only finitely many values of integers r such that at least one of $E_r^{p-r,q+r-1}$, $E_r^{p+r,q-r+1}$ is nonzero. Thus for r large enough, we have

$$E_r^{p-r,q+r-1} = 0 = E_r^{p+r,q-r+1} \ .$$

But then the differentials ending and starting at $E_r^{p,q}$ are zero, so passing to cohomology yields $E_r^{p,q} = E_{r+1}^{p,q}$ for all large enough integers r, which proves that the spectral sequence converges. □

Definition 5.1.7. Let $(E_r^{p,q})$ be a bounded spectral sequence of A-modules starting at the E_a-page for some integer a. Let $H^* = (H^n)_{n\in\mathbb{Z}}$ be a graded A-module (think: the cohomology of some cochain complex). We say that the spectral sequence $(E_r^{p,q})$ *converges to* H^* and write

$$E_a^{p,q} \Rightarrow H^{p+q}$$

if there exists a filtration of H^* by graded submodules F^pH^* such that $F^{p+1}H^n \subseteq F^pH^n$ for all integers p, n, and such that

$$E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$$

for all $(p,q) \in \mathbb{Z}^2$.

Remark 5.1.8. The convergence of a spectral sequence $E_a^{p,q} \Rightarrow H^{p+q}$ does not mean that we can determine H^n outright; what this says is that there is a filtration of H^n whose subquotients are isomorphic to the modules $E_\infty^{p,q}$, with (p,q) running over the set of all pairs of integers such that $p+q=n$. By the boundedness assumption, there are only finitely many such pairs. For instance, if one of the $E_\infty^{p,q}$ with $p+q=n$ is nonzero, then H^n is nonzero, and if k is a field such that the $E_\infty^{p,q}$ are finite-dimensional, then $\dim_k(H^n) = \sum_{p+q=n} \dim_k(E_\infty^{p,q})$.

Example 5.1.9. Let X be a cochain complex and Y a subcomplex of X. Consider X with the filtration $F^0X = X$, $F^1X = Y$, and $F^2X = \{0\}$ (the zero subcomplex of X). Consider $H^n = H^n(X)$ filtered with $F^0H^n = H^n$, $F^1H^n = \mathrm{Im}(H^n(Y) \to H^n(X))$, and $F^2H^n = \{0\}$. The connecting homomorphism $d^n : H^n(X/Y) \to H^{n+1}(Y)$ can be regarded as the differential of the E_1-page of a spectral sequence of the form

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^n(X/Y) & \xrightarrow{d^n} & H^{n+1}(Y) & \longrightarrow & 0 \\
 & & & & & & \\
0 & \longrightarrow & H^{n-1}(X/Y) & \xrightarrow{d^{n-1}} & H^n(Y) & \longrightarrow & 0 \\
 & & & & & & \\
0 & \longrightarrow & H^{n-2}(X/Y) & \xrightarrow{d^{n-2}} & H^{n-1}(Y) & \longrightarrow & 0
\end{array}$$

with the two nonzero columns in degree 0 and 1. The E_2-page is obtained from passing to cohomology, thus of the form

$$\begin{array}{cc}
\ker(d^n) & H^{n+1}(Y)/\mathrm{Im}(d^n) \\
 & \\
\ker(d^{n-1}) & H^n(Y)/\mathrm{Im}(d^{n-1}) \\
 & \\
\ker(d^{n-2}) & H^{n-1}(Y)/\mathrm{Im}(d^{n-2})
\end{array}$$

with zero differential; that is, $E_2 = E_\infty$. The fact that this spectral sequence converges to $H^* = H^*(X)$ is equivalent to the long exact cohomology sequence. Indeed, the convergence of this spectral sequence to H^* means that H^n is filtered by the the term $H^n(Y)/\mathrm{Im}(d^{n-1})$ in position $(1, n-1)$ and $\ker(d^n)$ in position $(0,n)$. But $\ker(d^n)$ is equal to the image of $H^n(X) \to H^n(X/Y)$ by the exactness of long cohomology sequence, and the map $H^n(Y) \to H^n(X)$ has image $H^n(Y)/\mathrm{Im}(d^{n-1})$.

5.2 The spectral sequence of a filtration

Any filtered complex gives rise to a spectral sequence starting at the E_1-page. A filtration of a cochain complex X by subcomplexes F^pX satisfying $F^{p+1}X \subseteq F^pX$ for $p \in \mathbb{Z}$ is called *bounded* if for any $n \in \mathbb{Z}$ there exist integers p and q such that $F^pX^n = X^n$ and $F^qX^n = \{0\}$; that is, in each fixed degree, the filtration induced by the subcomplexes F^pX is finite.

Theorem 5.2.1. *Let X be a cochain complex of A-modules with a bounded filtration by subcomplexes F^pX such that $F^{p+1}X \subseteq F^pX$ for $p \in \mathbb{Z}$. There is a bounded spectral sequence*

$$E_1^{p,q} \Rightarrow H^{p+q}(X)$$

with

$$E_1^{p,q} = H^{p+q}(F^pX/F^{p+1}X)$$

for any p ,$q \in \mathbb{Z}$, and

$$E_\infty^{p,q} = F^pH^{p+q}(X)/F^{p+1}H^{p+q}(X)$$

where $F^pH^n(X) = \mathrm{Im}(H^n(F^pX) \to H^n(X))$ with the map being induced by the inclusion $F^pX \subseteq X$, for any n, p, $q \in \mathbb{Z}$.

Proof. Denote by $\delta = (\delta^n)_{n\in\mathbb{Z}}$ the differential of X. We define for p, q, $r \in \mathbb{Z}$, $r \geq 1$, the following submodules of X^{p+q}.

$$Z_r^{p,q} = \{x \in F^pX^{p+q} \mid \delta^{p+q}(x) \in F^{p+r}X^{p+q+1}\}$$

In other words, $Z_r^{p,q}$ is the inverse image in $X^{p,q}$ of the differential in degree $p+q$ of the complex $F^pX/F^{p+r}X$.

$$B_r^{p,q} = \delta^{p+q-1}(F^{p-r}X^{p+q})$$
$$Z_\infty^{p,q} = \ker(\delta^{p+q}) \cap F^pX^{p+q}$$
$$B_\infty^{p,q} = \mathrm{Im}(\delta^{p+q-1}) \cap F^pX^{p+q} \ .$$

One verifies the following inclusions:

$$B_0^{p,q} \subseteq B_1^{p,q} \subseteq \cdots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdot \subseteq Z_1^{p,q} \subseteq Z_0^{p,q} \subseteq X^{p+q} \ .$$

We set

$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) \ ,$$

where p, q, $r \in \mathbb{Z}$ such that $r \geq 1$. The rest of the proof is a verification of the following statements:

(1) The differential δ of X induce a maps $E_r^{p,q} \to E_r^{p+r,q-r+1}$.

(2) Taking cohomology sends the page E_r^{**} to the subsequent page E_{r+1}^{**}.

(3) We have $E_1^{p,q} \cong H^{p+q}(F^pX/F^{p+1}X)$.

(4) We have $E_\infty^{p,q} \cong F^pH^{p+q}(X)/F^{p+1}H^{p+q}(X)$. □

We describe next one of the major construction principles which yield filtered complexes: the total complex of a double complex comes with two filtrations.

Definition 5.2.2. A double cochain complex of A-modules is a triple (X, δ, ϵ) consisting of
(a) a bigraded A-module $X^{**} = (X^{p,q})_{p,q \in \mathbb{Z}}$,
(b) a *horizontal differential* $\delta : X^{p,q} \to X^{p+1,q}$ of *bidegree* $(1, 0)$ satisfying $\delta^{p+1,q} \circ \delta^{p,q} = 0$ for all p, $q \in \mathbb{Z}$,
(c) a *vertical differential* $\epsilon : X^{p,q} \to X^{p,q+1}$ of *bidegree* $(0, 1)$ satisfying $\epsilon^{p,q+1} \circ \epsilon^{p,q} = 0$ for all p, $q \in \mathbb{Z}$,
with the property that

$$\epsilon^{p+1,q} \circ \delta^{p,q} = -\delta^{p,q+1} \circ \epsilon^{p,q} \ ;$$

that is, the squares

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
\longrightarrow & & X^{p,q+1} & \xrightarrow{\delta^{p,q+1}} & X^{p+1,q+1} & \longrightarrow & \\
 & & {\scriptstyle \epsilon^{p,q}} \uparrow & & \uparrow {\scriptstyle \epsilon^{p+1,q}} & & \\
\longrightarrow & & X^{p,q} & \xrightarrow[\delta^{p,q}]{} & X^{p+1,q} & \longrightarrow & \\
 & & \uparrow & & \uparrow & &
\end{array}$$

anticommute for all p, $q \in \mathbb{Z}$.

That is, a double complex can be regarded as a sequence of horizontal cochain complexes (with differentials given by δ), such that the vertical maps ϵ are 'cochain maps up to signs'. One verifies that then ϵ preserves $\ker(\delta)$ and $\mathrm{Im}(\delta)$, hence induces a vertical graded map ϵ on the 'horizontal' cohomology of the complexes with differential δ. Similarly, a double complex can be regarded as a sequence of vertical cochain complexes such that the horizontal maps are 'cochain maps up to signs'.

Definition 5.2.3. Let (X, δ, ϵ) be a double cochain complex of A-modules. The *total complex of* X is the cochain complex, denoted $\mathrm{tot}(X)$, defined by

$$\mathrm{tot}(X)^n = \oplus_{p+q=n} X^{p,q} = \oplus_{p \in \mathbb{Z}} X^{p,n-p} = \oplus_{q \in \mathbb{Z}} X^{n-q,q}$$

for $n \in \mathbb{Z}$, where in the first sum (p, q) runs over all $(p, q) \in \mathbb{Z}^2$ such that $p+q = n$, with differential

$$\Delta = \delta + \epsilon \ ;$$

explicitly, Δ^n is the sum of the maps $\delta^{p,q}$ and $\epsilon^{p,q}$, the sum taken over all $(p, q) \in \mathbb{Z}^2$ such that $p + q = n$.

One verifies that $\Delta \circ \Delta = 0$; this makes use of the anticommutativity of the differentials in the definition of double complexes. If for some $n \in \mathbb{Z}$ there are infinitely many pairs of integers (p, q) satisfying $p + q = n$, then it would make a difference whether we define $\mathrm{tot}(X)^n$ as direct sum or as direct product of the $X^{p,q}$ with $p + q = n$, and both versions may be useful, depending on circumstances. We will keep the focus on bounded spectral sequences, so this issue will not arise here. We describe next the two filtrations of the total complex of a double complex.

Definition 5.2.4. Let (X, δ, ϵ) be a double cochain complex. Set $Y = \mathrm{tot}(X)$. Define

$$F_I^p Y^n = \oplus_{r \geq p} X^{r,n-r} \ ,$$

$$F_{II}^p Y^n = \oplus_{r \geq p} X^{n-r,r} \ ,$$

for any n, $p \in \mathbb{Z}$.

An easy verification shows that $F_I^p Y$ and $F_{II}^p Y$ are subcomplexes of $Y = \mathrm{tot}(X)$, for any $p \in \mathbb{Z}$. As mentioned before, we can take the cohomology in a double complex (X, δ, ϵ) in two ways: either horizontally with respect to δ, or vertically, with respect to ϵ. We use the following notation.

Definition 5.2.5. Let (X, δ, ϵ) be a double cochain complex of A-modules. For any $(p, q) \in \mathbb{Z}^2$ we set

$$H_I^{p,q}(X) = \ker(\delta^{p,q})/\mathrm{Im}(\delta^{p-1,q}) \ ,$$

$$H_{II}^{p,q}(X) = \ker(\epsilon^{p,q})/\mathrm{Im}(\epsilon^{p,q-1}) \ .$$

We regard $H_I^{p,q}(X)$ again as a double complex, with horizontal differential now zero, while the vertical differential is induced by ϵ, and will be denoted by $\bar{\epsilon}$. Similarly, we regard $H_{II}^{p,q}(X)$ again as a double complex, with vertical differential zero and horizontal differential $\bar{\delta}$ induced by δ. Thus we can apply taking horizontal and vertical cohomology again; this leads to considering the bigraded objects (that is, double complexes in which both differentials are zero) of the form $H_{II}H_I(X)$ and $H_I H_{II}(X)$.

Theorem 5.2.6. *Let X be a first quadrant double cochain complex of A-modules; that is, $X^{p,q} = \{0\}$ if at least one of p or q is negative. There are two first quadrant spectral sequences of the form*

$$_I E_2^{p,q} = H_I^{p,q} H_{II}(X) \Rightarrow H^{p+q}(X)$$

$$_{II} E_2^{p,q} = H_{II}^{p,q} H_I(X) \Rightarrow H^{p+q}(X)$$

Proof. The two filtrations F_I and F_{II} of $Y = \mathrm{tot}(X)$ described in 5.2.4 give rise, by Theorem 5.2.1, to two spectral sequences starting at the E_1-page, of the form

$$_I E_1^{p,q} \Rightarrow H^{p+q}(Y)$$

with

$$_I E_1^{p,q} = H^{p+q}(F_I^p Y / F_I^{p+1} Y)$$

and similarly for F_{II}. One verifies that

$$_I E_1^{p,q} = H_{II}^{p,q}(X)$$

and that the differential at the page $_I E_1$ is equal to the differential $\bar{\delta}$ induced by δ. Thus taking cohomology with respect to $\bar{\delta}$ yields the page $_I E_2$, and that is by our notation the same as applying H_I. This yields the first spectral sequence, and the analogous argument with F_{II} yields the second spectral sequence. □

Theorem 5.2.7. *Let (X, δ) be a cochain complex of A-modules such that $X^n = \{0\}$ for $n < 0$. There exists a double cochain complex of the form*

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,2} & \xrightarrow{\delta^{0,2}} & I^{1,2} & \xrightarrow{\delta^{1,2}} & I^{2,2} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,1} & \xrightarrow{\delta^{0,1}} & I^{1,1} & \xrightarrow{\delta^{1,1}} & I^{2,1} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,0} & \xrightarrow{\delta^{0,0}} & I^{1,0} & \xrightarrow{\delta^{1,0}} & I^{2,0} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
X^0 & \xrightarrow[\delta^0]{} & X^1 & \xrightarrow[\delta^1]{} & X^2 & \longrightarrow &
\end{array}$$

such that the following hold.

(i) $I^{p,q}$ is injective for all $p, q \geq 0$.

(ii) $\mathrm{Im}(\delta^{n,*})$ *is an injective resolution of* $\mathrm{Im}(\delta^n)$*, for $n \geq 0$.*

(iii) $\ker(\delta^{n,*})/\mathrm{Im}(\delta^{n-1,*})$ *is an injective resolution of* $\ker(\delta^n)/\mathrm{Im}(\delta^{n-1}) = H^n(X)$*, for $n \geq 0$.*

(iv) If $X^n = \{0\}$ for some integer $n \geq 0$, then $I^{n,q} = \{0\}$ for all $q \geq 0$.

Proof. The proof uses the horseshoe lemma and constructs this resolution inductively. □

The double complex $I^{*,*}$ obtained from removing the bottom row, is called a *Cartan-Eilenberg resolution of X*. An easy argument using long exact sequences shows that then $\ker(\delta^{n,*})$ is also an injective resolution of $\ker(\delta^n)$.

Theorem 5.2.8 (Lyndon-Hochschild-Serre spectral sequence)**.** *Suppose that k is a field. Let G be a finite group, N a normal subgroup, and U a kG-module. There is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(G/N; H^q(N; U)) \Rightarrow H^{p+q}(G; U)$$

Proof. We have $H^n(G; U) = \mathrm{Ext}^n_{kG}(k; U)$. This can be calculated by choosing a projective resolution of k followed by applying the functor $\mathrm{Hom}_{kG}(-, U)$ to this resolution and then taking cohomology in degree n. It can also be calculated by choosing an injective resolution of U followed by applying the functor $\mathrm{Hom}_{kG}(k, -)$ to this resolution and then taking cohomology in degree n. It is a specialty of finite group algebras that their classes of projective and injective modules coincide (this follows from the fact that finite group algebras are *symmetric*, hence *self-injective*). Note that the functor $\mathrm{Hom}_{kG}(k, -)$ applied to the kG-module is the same as taking G-fixed points U^G in U; indeed, we have a natural isomorphism

$$\mathrm{Hom}_{kG}(k, U) \cong U^G$$

sending a kG-homomorphism $\tau : k \to U$ to $\tau(1)$. Applied to N instead of G, we have $\mathrm{Hom}_{kN}(k, U) \cong U^N$. Since N is normal in G, the action of G on U preserves U^N, and so U^N is again a kG-module. Moreover, N acts by definition trivially on U^N, so U^N inherits as kG/N-module structure, so that it makes sense to take G/N-fixed points in U^N. We clearly have

$$U^G = (U^N)^{G/N}$$

and this should be understood as the composition of two functors from $\mathrm{Mod}(kG)$ to $\mathrm{Mod}(k)$, namely

$$\mathrm{Hom}_{kG}(k, -) = \mathrm{Hom}_{kG/N}(k, -) \circ \mathrm{Hom}_{kN}(k, -)$$

The first of these two functors, sending a kG-module U to the kG/N-module U^N, has an important structural property: it preserves injectives, or equivalently, it preserves projectives. Indeed, the N-fixed points in the free kG-module kG of rank 1 are easily seen to be equal to $(\sum_{y \in N} y)kG \cong kG/N$.

With the preliminary observations, take an injective resolution of U; this yields an exact cochain complex

$$0 \longrightarrow U \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$$

To calculate $H^q(N; U)$, we need to apply the fixed point functor $\mathrm{Hom}_{kN}(k, -)$. This yields a cochain complex of kG/N-modules of the form

$$0 \longrightarrow U^N \longrightarrow (J^0)^N \longrightarrow (J^1)^N \longrightarrow \cdots$$

Note that all modules apart from U^N are again injective kG/N-modules. Construct a Cartan-Eilenberg resolution of this complex. Then apply the G/N-fixed point functor $\mathrm{Hom}_{kG/N}(k, -)$ to the entire Cartan-Eilenberg resolution. One ends up with a double cochain complex. This double complex yields two spectral sequences. One shows that one of these collapses (using the injectivity of the $(J^i)^N$), and uses this to show that the other of these converges to $H^{p+q}(G; U)$ as stated. □

The Lyndon-Hochschild-Serre spectral sequence is a crucial ingredient in the proof of the following result:

Theorem 5.2.9 (Evens-Venkov)**.** *Let G be a finite group and k a field. Then $H^*(G; k)$ is a finitely generated graded-commutative k-algebra.*

The construction of the Lyndon-Hochschild-Serre spectral sequence is a special case of what is known as a *Grothendieck spectral sequence*. These are obtained from playing of the derived functors of two composable functors and those of the composition of the two functors.

Definition 5.2.10. Let A, B be algebras over a commutative ring k. Let $\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ be a covariant functor and $n \geq 0$. The n-the *right derived functor* $R^n\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is defined by

$$R^n\mathcal{F}(U) = H^n(\mathcal{F}(I)),$$

where I is an injective resolution of U. The n-the *left derived functor* $L_n\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is defined by

$$L_n\mathcal{F}(U) = H_n(\mathcal{F}(P)),$$

where P is an projective resolution of U.

We have analogous definitions for contravariant functors.

Exercise 5.2.11. With the notation of 5.2.10, show that if $\mathcal{F}$ is left exact, then $R^0\mathcal{F} \cong \mathcal{F}$, and if $\mathcal{F}$ is right exact, then $L_0\mathcal{F} \cong \mathcal{F}$. Show that the functors $L_n\mathcal{F}$, $R^n\mathcal{F}$ are independent, up to unique isomorphism of functors, of the choices of resolutions.

Example 5.2.12. Let U, V be A-modules. The fact that $\mathrm{Ext}_A^n(U,V)$ can be calculated by using either a projective resolution of U or an injective resolution of V translates to the equality

$$\mathrm{Ext}_A^n(U,V) = L_n(\mathrm{Hom}_A(-,V))(U) = R^n(\mathrm{Hom}_A(U,-))(V) \ .$$

In particular, if G is a group and U a kG-module, then

$$H^n(G;U) = L_n(\mathrm{Hom}_{kG}(-,U))(k) = R^n(\mathrm{Hom}_{kG}(k,-))(U)$$

Definition 5.2.13. Let A, B be algebras over a commutative ring k. Let $\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ be a covariant functor. An A-module U is called *$\mathcal{F}$-acyclic* if $R^n\mathcal{F}(U) = \{0\}$ for all $n > 0$.

Any injective module is $\mathcal{F}$-acyclic for any functor $\mathcal{F}$. The point of this definition is that one can use resolutions by $\mathcal{F}$-acyclic modules rather than injective modules to calculate the derived functors of $\mathcal{F}$.

Theorem 5.2.14. *Let A, B be algebras over a commutative ring k. Let $\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ be a covariant functor. Let J be a resolution of an A-module U such that all terms of J are $\mathcal{F}$-acyclic. Then $R^n\mathcal{F}(U) \cong H^n(\mathcal{F}(I))$ for $n \geq 0$.*

This allows more flexibility when it comes to calculating derived functors - for instance, in sheaf theory, a *flabby sheaf* on a space X is $\Gamma(X,-)$-acyclic, and hence, in order to calculate sheaf cohomology, one may use resolutions by flabby sheaves rather than injective sheaves. The following result, describing *Grothendieck spectral sequences*, takes this into account.

Theorem 5.2.15 (Grothendieck, [4]). *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ be left exact functors. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injective objects and that $\mathcal{F}$ sends injective objects in $\mathcal{A}$ to $\mathcal{G}$-acyclic objects in $\mathcal{B}$. There is a first quadrant spectral sequence*

$$E_2^{p,q} = (R^p\mathcal{G})(R^q\mathcal{F}(X)) \Rightarrow R^{p+q}(\mathcal{G} \circ \mathcal{F})(X)$$

for any object X in $\mathcal{A}$.

Proof. This follows the pattern we have aleady encountered in the proof of the Lyndon-Hochschild-Serre spectral sequence. We start with an injective resolution I of X. We apply the functor $\mathcal{F}$ to this resolution. Note that the terms of $\mathcal{F}(I)$ are $\mathcal{G}$-acyclic. We then consider a Cartan-Eilenberg resolution of this complex in $\mathcal{B}$, and we apply the functor $\mathcal{G}$. This yields a double complex in $\mathcal{C}$. One of the two spectral sequences associated with this double complex collapses (this is where we use that the terms of $\mathcal{F}(I)$ are $\mathcal{G}$-acyclic), and the other takes the form as in the statement. □

The Lyndon-Hochschild-Serre spectral sequence is a Grothendieck spectral sequence with $\mathcal{A} = \mathrm{Mod}(kG)$, $\mathcal{B} = \mathrm{Mod}(kG/N)$, $\mathcal{C} = \mathrm{Mod}(k)$, $\mathcal{F} = \mathrm{Hom}_{kN}(k,-)$, and $\mathcal{G} = \mathrm{Hom}_{kG/N}(k,-)$. So is the following spectral sequence in sheaf cohomology, due to Leray, which is fundamental in algebraic geometry.

Theorem 5.2.16 (Leray)**.** *Let X, Y be topological spaces and $f : X \to Y$ a continous map. Let $\mathcal{F}$ be a sheaf on X. There is a spectral sequence*

$$E_2^{p,q} = H^p(Y; R^q f_*(\mathcal{F})) \Rightarrow H^{p+q}(X; \mathcal{F})$$

where $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is the direct image functor.

Proof. This is a Grothendieck spectral sequence with $\mathcal{A} = \mathrm{Sh}(X)$, $\mathcal{B} = \mathrm{Sh}(Y)$, $\mathcal{C} = \mathbf{Ab}$ (the category of abelian groups), $\mathcal{F} = f_*$, $\mathcal{G} = \Gamma(Y; -)$ (the global section functor on Y), using the fact that $\Gamma(X; -) = \Gamma(Y; -) \circ f_*$ and that f_* preserves injectives. □

Chapter 6

Appendix: Category theory

Category theory considers mathematical objects systematically together with the structure preserving maps between them, providing a unifying language for many different mathematical concepts to which homological methods can be applied. We review in this chapter the basic category theoretic vocabulary: *category, functor, natural transformation*, and *adjunction.*

6.1 Categories, functors, and natural transformations

A category $\mathcal{C}$ consists of three types of data: an *object class*, a *morphism class*, and information on how to compose morphisms, with a short list of properties one would expect any reasonable category of mathematical objects to have.

Definition 6.1.1. A category $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$, called the *class of objects of* $\mathcal{C}$, for any X, $Y \in \mathrm{Ob}(\mathcal{C})$ a class $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, called the *class of morphisms from* X *to* Y *in* $\mathcal{C}$, and, for any X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$ a map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z), \quad (f,g) \mapsto g \circ f \ ,$$

called the *composition map*, subject to the following properties.

(1) The classes $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, with X, $Y \in \mathrm{Ob}(\mathcal{C})$, are pairwise disjoint. Equivalently, any morphism f in $\mathcal{C}$ determines uniquely a pair (X,Y) of objects in $\mathcal{C}$ such that $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

(2) (Identity morphisms) For any object $X \in \mathrm{Ob}(\mathcal{C})$, there is a distinguished morphism $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, called *identity morphism of* X, such that for any object $Y \in \mathcal{C}$, any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and any $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ we have $f \circ \mathrm{Id}_X = f$ and $\mathrm{Id}_X \circ g = g$.

(3) (Associativity) For any X, Y, Z, $W \in \mathrm{Ob}(\mathcal{C})$ and any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, $h \in \mathrm{Hom}_{\mathcal{C}}(Z,W)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$; this is an equality of morphisms in $\mathrm{Hom}_{\mathcal{C}}(X,W)$.

We say that X is an object in a category $\mathcal{C}$ if $X \in \mathrm{Ob}(\mathcal{C})$. The objects of a category form in general a *class*, not necessarily a set. A category whose object and morphism classes are sets is called a *small category.* A category with the property that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a set for any two objects X, Y in $\mathcal{C}$ is called *locally small.* A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ between two objects X,

Y in a category $\mathcal{C}$ is typically denoted by $f : X \to Y$ or by $X \xrightarrow{f} Y$. Morphisms are also called *maps*, although one should note that the morphisms of a category may be abstractly defined and do not necessarily induce any maps in a set theoretic sense. We write $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$, and call the morphisms in $\mathrm{End}_{\mathcal{C}}(X)$ the *endomorphisms of* X. If $\mathrm{End}_{\mathcal{C}}(X)$ is a set, then $\mathrm{End}_{\mathcal{C}}(X)$ together with the composition of morphisms is a monoid with unit element Id_X.

Examples 6.1.2.

(1) We denote by **Sets** the category of sets, having as objects the class of sets and as morphisms arbitrary maps between sets. This is a locally small but not small category - considering the set of all sets leads to what is known as *Russell's paradox.*

(2) For k a field, we denote by $\mathbf{Vect}(k)$ the category of k-vector spaces; that is, the objects of $\mathbf{Vect}(k)$ are the k-vector spaces, and the morphisms are k-linear transformations between k-vector spaces. For U, V two k-vector spaces, we write $\mathrm{Hom}_k(U, V)$ instead of $\mathrm{Hom}_{\mathbf{Vect}(k)}(U, V)$ for the space of k-linear transformations from U to V, and we write $\mathrm{End}_k(U) = \mathrm{Hom}_k(U, U)$. Note that the sets $\mathrm{Hom}_k(U, V)$ are again k-vector spaces, not just sets, and that the composition maps are k-bilinear.

(3) We denote by **Grps** the category of groups, with groups as objects and group homomorphisms as morphisms.

(4) We denote by **Top** the category of topological spaces, with continuous maps as morphisms.

(5) If $\mathcal{C}$ is a small category with a single object E, then, as noted above, $\mathrm{Hom}_{\mathcal{C}}(E, E)$ is a monoid. Conversely, if M is a monoid, we can consider M as a the morphism set of a category $\mathbf{M}$ with a single object $*$, such that the morphism set in $\mathbf{M}$ from $*$ to $*$ is equal to M, and such that composition of morphisms in $\mathbf{M}$ is equal to the product in M.

(6) We denote by $\mathbf{Alg}(k)$ the category of k-algebras, with algebra homomorphisms as morphisms.

(7) For A an algebra over a commutative ring k, we denote by $\mathrm{Mod}(A)$ the category of left A-modules; that is, the objects of $\mathrm{Mod}(A)$ are the left A-modules, and morphisms are A-module homomorphisms. For U, V two A-modules we write $\mathrm{Hom}_A(U, V)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(U, V)$ for the set of A-homomorphisms from U to V. Similarly, we write $\mathrm{End}_A(U)$ instead of $\mathrm{End}_{\mathrm{Mod}(A)}(U)$. Note that if k is a field, then $\mathbf{Vect}(k) = \mathrm{Mod}(k)$ and $\mathbf{vect}(k) = \mathrm{mod}(k)$.

Definition 6.1.3. Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is defined by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ for all X, $Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, with composition $g \bullet f$ in $\mathcal{C}^{\mathrm{op}}$ defined by $g \bullet f = f \circ g$, for any X, Y, $Z \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z) = \mathrm{Hom}_{\mathcal{C}}(Z, Y)$, and where $f \circ g$ is the composition in $\mathcal{C}$.

Definition 6.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if $\mathrm{Ob}(\mathcal{D})$ is a subclass of $\mathrm{Ob}(\mathcal{C})$, and if for any X, Y in $\mathrm{Ob}(\mathcal{D})$, the class $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ is a subclass of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, such that for any X, Y, $Z \in \mathrm{Ob}(\mathcal{D})$, the composition map $\mathrm{Hom}_{\mathcal{D}}(X, Y) \times \mathrm{Hom}_{\mathcal{D}}(Y, Z) \to \mathrm{Hom}_{\mathcal{D}}(X, Z)$ in $\mathcal{D}$ is the restriction to the morphism class in $\mathcal{D}$ of the composition map in the morphism class of $\mathcal{C}$. We say that the subcategory $\mathcal{D}$ of $\mathcal{C}$ is a *full subcategory,* if for any X, $Y \in \mathrm{Ob}(\mathcal{D})$ we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Examples 6.1.5.

(1) For k a field we denote by $\mathbf{vect}(k)$ the full subcategory of $\mathbf{Vect}(k)$ consisting of all finite-dimensional k-vector spaces.

(2) The category **grps** of finite groups is a full subcategory of the category of all groups **Grps**.

(3) The category of finitely generated left A-modules, denoted $\mathrm{mod}(A)$, is a full subcategory of $\mathrm{Mod}(A)$.

The philosophy of considering any mathematical object together with its structure preserving maps applies to categories as well. Functors are 'morphisms' between categories.

Definition 6.1.6. Let $\mathcal{C}$, $\mathcal{D}$ be categories. A *functor* or *covariant functor* $\mathcal{F}$ *from* $\mathcal{C}$ *to* $\mathcal{D}$ is a map $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ together with a family of maps, abusively all denoted by the same letter $\mathcal{F}$, from $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ to $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X),\mathcal{F}(Y))$ for all X, $Y \in \mathrm{Ob}(\mathcal{C})$, with the following properties.

(a) For all objects X in $\mathrm{Ob}(\mathcal{C})$ we have $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$.

(b) For all objects X, Y, Z in $\mathrm{Ob}(\mathcal{C})$ and morphisms $\varphi : X \to Y$ and $\psi : Y \to Z$ we have

$$\mathcal{F}(\psi \circ \varphi) = \mathcal{F}(\psi) \circ \mathcal{F}(\varphi) \ .$$

Similarly, a *contravariant functor from* $\mathcal{C}$ *to* $\mathcal{D}$ is map $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ together with a family of maps $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(Y),\mathcal{F}(X))$ for all X, $Y \in \mathrm{Ob}(\mathcal{C})$, with the following properties.

(c) For all objects X in $\mathrm{Ob}(\mathcal{C})$ we have $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$.

(d) For all objects X, Y, Z in $\mathrm{Ob}(\mathcal{C})$ and morphisms $\varphi : X \to Y$ and $\psi : Y \to Z$ we have

$$\mathcal{F}(\psi \circ \varphi) = \mathcal{F}(\varphi) \circ \mathcal{F}(\psi) \ .$$

Equivalently, a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Functors can be composed in the obvious way, by composing the maps on objects and on morphisms. Composing a covariant functor with a contravariant functor (in either order) yields a contravariant functor. Composing two contravariant functors yields a covariant functor. On every catagory $\mathcal{C}$ there is the identity functor $\mathrm{Id}_{\mathcal{C}}$ which is the identity map on $\mathrm{Ob}(\mathcal{C})$ and the family of identity maps on the morphism sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, X, $Y \in \mathrm{Ob}(\mathcal{C})$. Since the object classes of categories need not be sets, we cannot consider the category having all categories as objects and functors as morphisms. We can though consider the category **Cat** having as objects small categories and as morphisms all functors between small categories; that is, for two small categories $\mathcal{C}$, $\mathcal{D}$, we denote by $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C},\mathcal{D})$ the set of functors from $\mathcal{C}$ to $\mathcal{D}$.

Examples 6.1.7.

(1) There is a class of trivial functors, called *forgetful functors*, obtained from ignoring a part of the structure of a mathematical object. For instance, we have a forgetful functor $\mathbf{Alg}(k) \to \mathbf{Vect}(k)$ which sends a k-algebra to its underlying k-vector space (that is, we ignore the multiplication in the algebra). Every k-vector space is in particular an abelian group, so this yields a forgetful functor $\mathbf{Vect}(k) \to \mathbf{Ab}$ sending a vector space to the underlying abelian group (that is, we ignore the scalar muliplication). Every abelian group is in particular a set, so we get a forgetful functor $\mathbf{Ab} \to \mathbf{Sets}$.

(2) There is a functor from **Grps** to $\mathbf{Alg}(k)$ sending a group G to the group algebra kG and sending a group homomorphism $\varphi : G \to H$ to the algebra homomorphism $kG \to kH$ obtained by extending φ linearly. There is also a functor $\mathbf{Alg}(k) \to \mathbf{Grps}$ sending a k-algebra A to the group of invertible elements $A^\times$. To see that this is functorial, one verifies that an algebra homomorphism $\alpha : A \to B$ sends $A^\times$ to $B^\times$, hence induces a group homomorphism $A^\times \to B^\times$.

(3) There is a class of functors called *representable functors.* Let $\mathcal{C}$ be a category such that for any two objects X, X' the class $\mathrm{Hom}_\mathcal{C}(X, X')$ is a set. Fix an an object X in $\mathcal{C}$. We define a functor $\mathrm{Hom}_\mathcal{C}(X, -)$ from $\mathcal{C}$ to the category of sets as follows. For any object Y in $\mathcal{C}$, the functor $\mathrm{Hom}_\mathcal{C}(X, -)$ sends Y to the set $\mathrm{Hom}_\mathcal{C}(X, Y)$. For any morphism $f : Y \to Z$ in $\mathcal{C}$ the functor $\mathrm{Hom}_\mathcal{C}(X, -)$ sends f to the map, denoted $\mathrm{Hom}_\mathcal{C}(X, f)$ which is induced by composition with f; that is, which sends $h \in \mathrm{Hom}_\mathcal{C}(X, Y)$ to $f \circ h \in \mathrm{Hom}_\mathcal{C}(X, Z)$. One easily sees that this is a functor. This construction applied to $\mathcal{C}^{\mathrm{op}}$ yields also a contravariant functor $\mathrm{Hom}_\mathcal{C}(-, X)$, sending Y to $\mathrm{Hom}_\mathcal{C}(Y, X)$ and sending f to the map denoted $\mathrm{Hom}_\mathcal{C}(f, X)$ induced by precomposition with f; that is, $\mathrm{Hom}_\mathcal{C}(f, X)$ sends $h \in \mathrm{Hom}_\mathcal{C}(Z, X)$ to $h \circ f \in \mathrm{Hom}_\mathcal{C}(Z, Y)$. Functors of this form are called *representable.* If we consider both X and Y as variables, then $\mathrm{Hom}_\mathcal{C}(-, -)$ is what we call a *bifunctor* . Depending on what additional structures the category $\mathcal{C}$ has, the representable functors may have as target category not just the category of sets but categories with more structure. For instance, if A is a k-algebra and U an A-module, then the representable functor $\mathrm{Hom}_A(U, -)$ and its contravariant analogue $\mathrm{Hom}_A(-, U)$ are functors from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(k)$.

(4) Let A, B be k-algebras, and let M be an A-B-bimodule. There is a functor $M \otimes_B -$ from $\mathrm{Mod}(B)$ to $\mathrm{Mod}(A)$ sending a B-module V to the A-module $M \otimes_B V$ and a B-homomorphism $\psi : V \to V'$ to the A-homomorphism $\mathrm{Id}_M \otimes \psi : M \otimes_B V \to M \otimes_B V'$. There is a similar functor $- \otimes_A M$ from $\mathrm{Mod}(A^{\mathrm{op}})$ to $\mathrm{Mod}(B^{\mathrm{op}})$. There is a functor $\mathrm{Hom}_A(M, -)$ from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(B)$, sending an A-module U to $\mathrm{Hom}_A(M, U)$, viewed as a B-module via $(b \cdot \varphi)(m) = \varphi(mb)$, where $\varphi \in \mathrm{Hom}_A(M, U)$, $m \in M$, $b \in B$. There is a similar functor $\mathrm{Hom}_{B^{\mathrm{op}}}(M, -)$ from $\mathrm{Mod}(B^{\mathrm{op}})$ to $\mathrm{Mod}(A^{\mathrm{op}})$.

Pushing our philosophy of considering mathematical objects with their structural maps even further, we view now functors as objects and define morphisms between functors as follows.

Definition 6.1.8. Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let $\mathcal{F}$, $\mathcal{F}'$ be functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation from $\mathcal{F}$ to $\mathcal{F}'$* is a family $\varphi = (\varphi(X))_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms $\varphi(X) \in \mathrm{Hom}_\mathcal{D}(\mathcal{F}(X), \mathcal{F}'(X))$ such that for any morphism $f : X \to Y$ in $\mathcal{C}$ we have $\mathcal{F}'(f) \circ \varphi(X) = \varphi(Y) \circ \mathcal{F}(f)$; that is, we have a commutative diagram of morphisms in the category $\mathcal{D}$ of the form

$$
\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{\varphi(X)} & \mathcal{F}'(X) \\
{\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}'(f)} \\
\mathcal{F}(Y) & \xrightarrow[\varphi(Y)]{} & \mathcal{F}'(Y)
\end{array}
$$

By considering contravariant functors from $\mathcal{C}$ to $\mathcal{D}$ as convariant functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$ we get an abvious notion of natural transformation between contravariant functors from $\mathcal{C}$ to $\mathcal{D}$.

Every functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ gives rise to the *identity transformation* $\mathrm{Id}_\mathcal{F} : \mathcal{F} \to \mathcal{F}$ consisting of the family of identity morphisms $\mathrm{Id}_{\mathcal{F}(X)}$, $X \in \mathrm{Ob}(\mathcal{C})$. Natural transformations can be composed: if $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$ are functors from $\mathcal{C}$ to $\mathcal{D}$ and $\varphi : \mathcal{F} \to \mathcal{F}'$, $\psi : \mathcal{F}' \to \mathcal{F}''$ are natural transformations, then the family $\psi \circ \varphi$ of morphisms $\psi(X) \circ \varphi(X) : \mathcal{F}(X) \to \mathcal{F}''(X)$ is a natural transformation from $\mathcal{F}$ to $\mathcal{F}''$, and this composition of natural transformations is associative. As in the case of the category of categories there are set theoretic issues if we consider the category of functors from $\mathcal{C}$ to $\mathcal{D}$ with natural transformations as morphisms. If we assume that $\mathcal{C}$ is small, then the functors from $\mathcal{C}$ to an arbitrary category $\mathcal{D}$, together with natural transformations as morphisms, form a category. There is an obvious extension of the natural transformation to bifunctors.

Examples 6.1.9. **(1)** Let $\mathcal{C}$ be a category, X, X' objects, and let $\varphi : X \to X'$ be a morphism in $\mathcal{C}$. Then φ induces a natural transformation from $\mathrm{Hom}_{\mathcal{C}}(X', -)$ to $\mathrm{Hom}_{\mathcal{C}}(X, -)$, given by the family of maps $\mathrm{Hom}_{\mathcal{C}}(X', Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ sending $\tau \in \mathrm{Hom}_{\mathcal{C}}(X', Y)$ to $\tau \circ \varphi$, and φ induces a natural transformation from $\mathrm{Hom}_{\mathcal{C}}(-, X)$ to $\mathrm{Hom}_{\mathcal{C}}(-, X')$ sending $\tau \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ to $\varphi \circ \tau$, for all objects Y in $\mathcal{C}$.

(2) Let A, B be k-algebras and let M, M' be A-B-bimodules. Any bimodule homomorphism $\alpha : M \to M'$ induces a natural transformation from $M \otimes_B -$ to $M' \otimes_B -$ given by the family of maps $\alpha \otimes \mathrm{Id}_V : M \otimes_B V \to M' \otimes_B V$ for all B-modules V. Similarly, any such α induces a natural transformation from $\mathrm{Hom}_A(M', -)$ to $\mathrm{Hom}_A(M, -)$, as in the previous example.

Definition 6.1.10. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Two functors $\mathcal{F}$, $\mathcal{F}'$ from $\mathcal{C}$ to $\mathcal{D}$ are called *isomorphic* if there are natural transformations $\varphi : \mathcal{F} \to \mathcal{F}'$ and $\psi : \mathcal{F}' \to \mathcal{F}$ such that $\psi \circ \varphi = \mathrm{Id}_{\mathcal{F}}$ and $\varphi \circ \psi = \mathrm{Id}_{\mathcal{F}'}$.

If $\varphi : \mathcal{F} \to \mathcal{F}'$ is a natural transformation such that all morphisms $\varphi(X) : \mathcal{F}(X) \to \mathcal{F}'(X)$ are isomorphisms, then the family of morphisms $\psi(X) = \varphi(X)^{-1}$ is a natural transformation from $\mathcal{F}'$ to $\mathcal{F}$ satisfying $\psi \circ \varphi = \mathrm{Id}_{\mathcal{F}}$ and $\varphi \circ \psi = \mathrm{Id}_{\mathcal{F}'}$.

Definition 6.1.11. Two categories $\mathcal{C}$ and $\mathcal{D}$ are called *equivalent* if there are functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{F}$ such that $\mathcal{G} \circ \mathcal{F} \cong \mathrm{Id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{Id}_{\mathcal{D}}$, and the functors $\mathcal{F}$, $\mathcal{G}$ arising in this way are called *equivalences of categories.*

Thus an equivalence $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ need not induce a bijection between $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ob}(\mathcal{D})$, but it induces a bijection between the isomorphism classes in $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ob}(\mathcal{D})$.

6.2 Epimorphisms, monomorphisms, kernels and cokernels

Morphisms in a category are abstract mathematical objects and need not be maps between sets. One of the challenges is to extend to morphisms some standard notions of maps such as the property of being injective or surjective, without referring to elements in objects. The category theoretic version of surjective and injective maps are as follows.

Definition 6.2.1. Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. The morphism f is called an *epimorphism* if for any two morphisms g, g' from Y to any other object Z satisfying $g \circ f = g' \circ f$ we have $g = g'$. The morphism f is called a *monomorphism* if for any two morphisms g, g' from any other object Z to X satisfying $f \circ g = f \circ g'$ we have $g = g'$. The morphism f is called an *isomorphism* if there exists a morphism $h : Y \to X$ satisfying $h \circ f = \mathrm{Id}_X$ and $f \circ h = \mathrm{Id}_Y$. An isomorphism which is an endomorphism of an object X is called an *automorphism* of X.

There are various ways to reformulate this definition. For instance, $f : X \to Y$ is an epimorphism, if and only if for any object Z the map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ sending $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$

to $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is injective. Similarly, $f : X \to Y$ is a monomorphism, if and only if for any object W the map $\mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(W, Y)$ sending $g \in \mathrm{Hom}_{\mathcal{C}}(W, X)$ to $f \circ g \in \mathrm{Hom}_{\mathcal{C}}(W, Y)$ is injective.

Exercise 6.2.2. Show that in the category of sets, the monomorphisms are the injective maps and the epimorphisms are the surjective maps. Show that in the category $\mathrm{Mod}(A)$ of modules over an algebra A the monomorphisms are the injective A-module homomorphisms and the epimorphisms are the surjective A-module homomorphisms. Show that in the category $\mathrm{Ch}(\mathrm{Mod}(A))$ the monomorphisms (resp. epimorphisms) are the chain maps which are injective (resp. surjective) A-homomorphisms in each degree.

Exercise 6.2.3. Show that the composition of two monomorphisms in a category is a monomorphism, and that the composition of two epimorphisms is an epimorphism.

Exercise 6.2.4. Show that if $f : X \to Y$ is an isomorphism in a category $\mathcal{C}$, then there is a *unique* morphism $h \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ satisfying $h \circ f = \mathrm{Id}_X$ and $f \circ h = \mathrm{Id}_Y$. The morphism h is then called the *inverse of* f and denoted by f^{-1}. Show that if $f : X \to Y$ and $g : Y \to Z$ are isomorphisms, then $g \circ f$ is an isomorphism with inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Show that the automorphisms of X in $\mathcal{C}$ form a subgroup $\mathrm{Aut}_{\mathcal{C}}(X)$ of the monoid $\mathrm{End}_{\mathcal{C}}(X)$.

Exercise 6.2.5. Show that a morphism f in a category $\mathcal{C}$ is a monomorphism if and only if f is an epimorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.

Exercise 6.2.6. Show that if f is an isomorphism in a category $\mathcal{C}$, then f is both a monomorphism and an epimorphism in $\mathcal{C}$.

The converse of the statement in the last exercise does not hold in general: there are examples of categories in which a morphism is both an epimorphism and a monomorphism but not an isomorphism.

Exercise 6.2.7. Let $\mathcal{C}$ be the category having the abelian group $\mathbb{Z}$ as unique object, with all abelian group endomorphisms of $\mathbb{Z}$ as morphisms. Show that every nonzero endomorphism of $\mathbb{Z}$ is both a monomorphism and an epimorphism in $\mathcal{C}$. Use this to give an example of a morphism which is both a monomorphism and an epimorphism, but not an isomorphism.

Definition 6.2.8. Let $\mathcal{C}$ be a category. An object E is *initial* if for every object Y in $\mathcal{C}$ there is a unique morphism $E \to Y$ in $\mathcal{C}$. An object T is *terminal* if for every object Y in $\mathcal{C}$ there is a unique morphism $Y \to T$ in $\mathcal{C}$. A *zero object* is an object which is both initial and terminal. If O is a zero object in $\mathcal{C}$ and $f : X \to Y$ a morphism in $\mathcal{C}$ such that $f = h \circ g$, where $g : X \to O$ and $h : O \to Y$ are the unique morphisms, then f is called a *zero morphism* in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

The identity morphism of an initial or terminal object is its only endomorphism, and there is exactly one morphism between any two initial or terminal objects, and hence any such morphism is an isomorphism. Thus if a category has an initial or terminal or zero object, such an object is unique up to unique isomorphism. As a consequence, if $\mathcal{C}$ has a zero object, then for any two objects X, Y in $\mathcal{C}$ there is exactly one zero morphism in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Composing the zero morphism with any other morphism yields again the zero morphism.

Examples 6.2.9.

(1) The category **Vect**(k) of vector spaces over a field k has the zero space $\{0\}$ as zero object. Similarly, for A an algebra over a commutative ring, the category $\text{Mod}(A)$ has the zero module $\{0\}$ as zero object.

(2) The category of groups **Grps** has the trivial group $\{1\}$ as zero object.

(3) The category of rings has $\mathbb{Z}$ has initial object: for any ring R there is a unique ring homomorphism $\mathbb{Z} \to R$ sending a positive integer n to $n \cdot 1_R = 1_R + 1_R + \cdots + 1_R$ (the sum of 1_R with itself n times); this is extended to Z by mapping 0 to 0_R and $-n$ to $-(n \cdot 1_R)$. Note though that Z is not a terminal object: for instance, there is no ring homomorphism from $\mathbb{Q}$ to $\mathbb{Z}$.

(4) The space with a single element, denoted $\{*\}$, is terminal in the category of topological spaces **Top**, but not initial.

Definition 6.2.10. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$ with a zero object. A *kernel of* f is a pair consisting of an object in $\mathcal{C}$, denoted $\ker(f)$, and a morphism $i : \ker(f) \to X$, such that $f \circ i = 0$ and such that for any object Z and any morphism $g : Z \to X$ satisfying $f \circ g = 0$ there is a unique morphism $h : Z \to \ker(f)$ satisfying $g = i \circ h$. Dually, a *cokernel of* f is a pair consisting of an object in $\mathcal{C}$, denoted $\text{coker}(f)$, and a morphism $p : Y \to \text{coker}(f)$, such that $p \circ f = 0$ and such that for any object Z and any morphism $g : Y \to Z$ satisfying $g \circ f = 0$ there is a unique morphism $h : \text{coker}(f) \to Z$ satisfying $g = h \circ p$.

The uniqueness properties in this definition imply that i is a monomorphism, p is an epimorphism, and the pairs $(\ker(f), i)$ and $(\text{coker}(f), p)$, if they exist, are unique up to unique isomorphism. A kernel becomes a cokernel in the opposite category, and vice versa.

6.3 Projective and injective objects

Informally, an object P in a category $\mathcal{C}$ is *projective* if every morphism starting at P lifts through any epimorphism, and an object I in $\mathcal{C}$ is *injective* if every morphism ending at I extends through any monomorphism. The precise definition is as follows.

Definition 6.3.1. Let $\mathcal{C}$ be a category. An object P in $\mathcal{C}$ is called *projective* if for any epimorphism $h : X \to Y$ and any morphism $g : P \to Y$ there is a morphism $f : P \to X$ such that $h \circ f = g$. An object I in $\mathcal{C}$ is called *injective* if for any monomorphism $h : X \to Y$ and any morphism $g : X \to I$ there is a morphism $f : Y \to I$ such that $f \circ h = g$.

Exercise 6.3.2. Show that an object P in a category $\mathcal{C}$ is projective (resp. injective) if and only it is injective (resp. projective) as an object in the opposite category $\mathcal{C}^{\text{op}}$.

Exercise 6.3.3. . Let A be an algebra over some commutative ring and P an A-module. Show that P is projective if and only if for any surjective A-homomorphism $\varphi : U \to V$ the induced k-linear map $\text{Hom}_A(P, U) \to \text{Hom}_A(P, V)$ sending $\alpha \in \text{Hom}_A(P, U)$ to $\varphi \circ \alpha$ is surjective.

Let A be a algebra over a commutative ring k and F and A-module, where we adopt the convention that a module is a unital left module, unless stated otherwise. A subset X of F is called a *basis of* F, if every element in F can be written uniquely in the form $\sum_{x \in X} a_x x$ with elements $a_x \in A$ of which only finitely many are nonzero. An A-module F is called *free* if it has a basis. If F is a free A-module and X a basis of F, then $F = \oplus_{x\in X} Ax$, and $Ax \cong A$ as a left module, for each $x \in A$. In other words, an A-module F is free if and only if F is isomorphic to a direct sum of (possibly infinitely many) copies of A.

Exercise 6.3.4. Let A be an algebra over some commutative ring. Show that any free A-module is projective and that any direct summand of a projective A-module is projective.

Exercise 6.3.5. Let i be an idempotent in a ring A; that is, $i^2 = i \neq 0$. Show that the left ideal Ai generated by i is a projective A-module. (*Hint*: show that $A(1-i)$ is a complement of Ai in A as a left A-module).

Projective modules of an algebra can be characterised as direct summands of free modules; this rewords earlier exercises.

Theorem 6.3.6. *Let A be a algebra over a commutative ring k, and let P be an A-module. The following are equivalent.*

(i) The A-module P is a projective object in the category $\mathrm{Mod}(A)$ of A-modules.

(ii) Any surjective A-homomorphism $\pi : U \to P$ from some A-module U to P splits; that is, there is an A-homomorphism $\sigma : P \to U$ such that $\pi \circ \sigma = \mathrm{Id}_P$.

(iii) The functor $\mathrm{Hom}_A(P, -) : \mathrm{Mod}(A) \to \mathrm{Mod}(k)$ is exact; that is, it sends any short exact sequence of A-modules to a short exact sequence of k-modules.

(iv) The module P is isomorphic to a direct summand of a free A-module.

Proof. We will use the fact from Exercise 6.2.2 that the epimorphisms in $\mathrm{Mod}(A)$ are the surjective A-homomorphisms and that the monomorphisms in $\mathrm{Mod}(A)$ are the injective A-homomorphisms. Suppose that (i) holds; that is, P is projective in $\mathrm{Mod}(A)$. Let $\pi : U \to P$ be a surjective A-homomorphism, where U is an A-module. Then in particular the identity map Id_P lifts through the surjective map π; that is, there is an A-homomorphism $\sigma : P \to U$ satisfying $\pi\circ\sigma = \mathrm{Id}_P$. Thus π splits. This shows that (i) implies (ii). Suppose that (ii) holds. Let X be any subset of P which generates P; that is, every element in P can be written in the form $\sum_{x\in X} a_x x$ for some $a_x \in A$ of which only finitely many are nonzero. Take for F a free A-module having a basis $\{e_x \mid x \in X\}$ indexed by X. That is, $F = \oplus_{x\in X} Ae_x$. Since F is free, there is a unique A-module homomorphism $\pi : F \to P$ sending e_x to x. This homomorphism is surjective since X generates P as an A-module. Thus, if (ii) holds, then π splits, showing that P is isomorphic to a direct summand of F. This shows that (ii) implies (iv). The equivalence of (i) and (iii) follows easily using the Exercise 6.3.3. It follows from Exercise 6.3.4 that (iv) implies (i). □

Exercise 6.3.7. Let A be an algebra over some commutative ring. Show that every A-module is isomorphic to a quotient of a free A-module.

Except for the characterisation of projective modules as direct summands of free modules, we have a similar result for injective modules.

Theorem 6.3.8. *Let A be a k-algebra and let I be an A-module. The following are equivalent:*

(i) The A-module I is an injective object in $\mathrm{Mod}(A)$.

(ii) Any injective A-homomorphism $\iota : I \to V$ from I to some A-module V splits; that is, there is an A-homomorphism $\kappa : V \to I$ such that $\kappa \circ \iota = \mathrm{Id}_I$.

(iii) The contravariant functor $\mathrm{Hom}_A(-, I) : \mathrm{Mod}(A) \to \mathrm{Mod}(k)$ is exact.

Exercise 6.3.9. Give the details of the proof of Theorem 6.3.8.

Exercise 6.3.10. . Let A be an algebra over some commutative ring and I an A-module. Show that I is injective if and only if for any injective A-homomorphism $\varphi : U \to V$ the induced k-linear map $\mathrm{Hom}_A(V, I) \to \mathrm{Hom}_A(U, I)$ sending $\alpha \in \mathrm{Hom}_A(V, I)$ to $\alpha \circ \varphi$ is surjective.

Exercise 6.3.11. Show that the additive group of rational numbers $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. Show that the additive quotient $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

6.4 Products and coproducts

Definition 6.4.1. Let $\mathcal{C}$ be a category, and let $\{X_j\}_{j\in I}$ be a familiy of objects in $\mathcal{C}$, where I is an indexing set. A *product of the family of objects* $\{X_j\}_{j\in I}$ is an object in $\mathcal{C}$, denoted $\prod_{j\in I} X_j$, together with a family of morphisms $\pi_i : \prod_{j\in I} X_j \to X_i$ for each $i \in I$, satisfying the following universal property: for any object Y in $\mathcal{C}$ and any family of morphisms $\varphi_i : Y \to X_i$, with $i \in I$, there is a unique morphism $\alpha : Y \to \prod_{j\in I} X_j$ satisfying $\varphi_i = \pi_i \circ \alpha$ for all $i \in I$.

The uniqueness of α implies that the product, if it exists at all, is uniquely determined up to unique isomorphism. By reversing the direction of morphisms, one obtains coproducts or direct sums.

Definition 6.4.2. Let $\mathcal{C}$ be a category, and let $\{X_j\}_{j\in I}$ be a familiy of objects in $\mathcal{C}$, where I is an indexing set. A *coproduct or direct sum of the family of objects* $\{X_j\}_{j\in I}$ is an object in $\mathcal{C}$, denoted $\coprod_{j\in I} X_j$, together with a family of morphisms $\iota_i : X_i \to \coprod_{j\in I} X_j$ for each $i \in I$, satisfying the following universal property: for any object Y in $\mathcal{C}$ and any family of morphisms $\varphi_i : X_i \to Y$, with $i \in I$, there is a unique morphism $\alpha : \coprod_{j\in I} X_j \to Y$ satisfying $\varphi_i = \alpha \circ \iota_i$ for all $i \in I$.

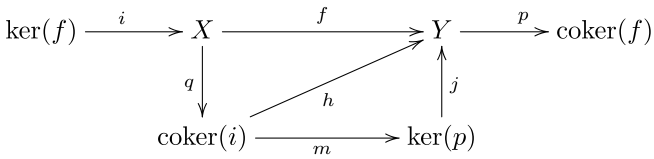
Definition 6.4.3. A category $\mathcal{C}$ with a zero object is called *additive* if the morphism classes $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are abelian groups, such that the composition of morphisms is biadditive, and such that coproducts of finite families of objects exist. A category $\mathcal{C}$ with a zero object is called *k-linear* if the morphism classes $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are k-vector spaces, such that the composition of morphisms is bilinear, and such that coproducts of finite families of objects exist.

Remark 6.4.4. In an additive or k-linear category we also have products of finite families, and products and coproducts of finite families of objects are isomorphic. To see this, let I be a finite indexing set and let $\{X_i\}_{i\in I}$ be a finite family of objects in an additive category $\mathcal{C}$. In order to simplify notation, we write $\coprod$ instead of $\coprod_{j\in I}$. We need to construct morphisms $\coprod X_j \to X_i$ for any $i \in I$ satisfying the universal property as in the definition of the product of the X_i. Let $i \in I$. For $j \in I$, denote by $\varphi : X_i \to X_j$ the morphism Id_{X_i} if $i = j$, and the zero morphism if $i \neq j$. The universal property of the coproduct yields a unique morphism $\pi_i : X_i \to \coprod X_j$ with the property $\pi_i \circ \iota_i = \mathrm{Id}_{X_i}$ and $\pi_j \circ \iota_i = 0$, where $i, j \in I$, $i \neq j$. To see that $\coprod X_j$, together with the morphisms $\pi_i : \coprod X_j \to X_i$, is a product, we consider a family of morphisms $\psi_i : Y \to X_i$, for $i \in I$, where Y is some object in $\mathcal{C}$. Then $\alpha = \sum_{j\in I} \iota_j \circ \psi_j$ is a morphism from $Y \to \coprod X_j$; this is well-defined since I is finite. Thus $\pi_i \circ \alpha = \sum_{j\in I} \pi_i \circ \iota_j \circ \psi_j = \psi_i$ for all $i \in I$. To see the uniqueness of α with this property, note first that the endomorphism $\gamma = \sum_{j\in I} \iota_j \circ \pi_j$ of $\coprod X_j$ satisfies $\gamma \circ \iota_i = \iota_i$ for all $i \in I$. But the identity morphism of $\coprod X_j$ is the unique endomorphism with this propery, where we use the universal property of coproducts. Thus γ is equal to the identity on $\coprod X_j$. Therefore, if

$\beta : Y \to \coprod X_j$ is any other morphism satisfying $\pi_i \circ \beta = \psi_i$ for all $i \in I$, then $\beta = \sum_{j \in I} \iota_j \circ \pi_j \circ \beta = \sum_{j \in J} \iota_j \circ \psi_j = \alpha$, which shows the uniqueness of α. This proves that $\coprod X_j$, together with the family of morphisms $\pi_i : \coprod X_j \to X_i$, with $i \in I$, is indeed product of the family $\{X_i\}_{i \in I}$.

6.5 Abelian categories

Module categories are additive, but they have more structure: all morphisms have kernels and cokernels, and there are isomorphism theorems relating kernels and images. Consider a k-algebra A and a homomorphism of A-mdoules $\varphi : U \to V$. Then $U/\ker(\varphi)$ is obtained by first taking the kernel $\ker(\varphi)$ and then taking the cokernel of the inclusion $\ker(\varphi) \subseteq U$. The image $\mathrm{Im}(\varphi)$ is obtained by first taking the cokernel $V \to \mathrm{coker}(\varphi) = V/\mathrm{Im}(\varphi)$, and then $\mathrm{Im}(\varphi)$ is the kernel of the map $V \to \mathrm{coker}(\varphi)$. The isomorphism theorem $U/\ker(\varphi) \cong \mathrm{Im}(\varphi)$ amounts therefore to stating that taking kernels and cokernels 'commute' in a canonical way. These considerations can be extended to additive categories. If $\mathcal{C}$ is an additive category, then for any morphism $f : X \to Y$ in $\mathcal{C}$ which has a kernel $i : \ker(f) \to X$ and a cokernel $p : Y \to \mathrm{coker}(f)$ there is a canonical morphism $\mathrm{coker}(i) \to \ker(p)$. This morphism is constructed as follows. Taking the cokernel of i yields an epimorphism $q : X \to \mathrm{coker}(i)$, and taking the kernel of p yields a monomorphism $j : \ker(p) \to Y$. Since $f \circ i = 0$, the definition of $\mathrm{coker}(i)$ yields a uniqe morphism $h : \mathrm{coker}(i) \to Y$ such that $h \circ q = f$. Then $0 = p \circ f = p \circ h \circ q$. Since q is an epimorphism, this implies that $p \circ h = 0$. Then the definition of $\ker(p)$ yields a unique morphism $m : \mathrm{coker}(i) \to \ker(p)$ satisfying $j \circ m = h$.



Definition 6.5.1. An additive category $\mathcal{C}$ is called an *abelian category* if for every morphism $f : X \to Y$ there exists a kernel $i : \ker(f) \to X$ and a cokernel $p : Y \to \mathrm{coker}(f)$, and if the canonical morphism $\mathrm{coker}(i) \to \ker(p)$ is an isomorphism.

Slightly abusing notation, one can rephrase the last condition as requiring the canonical morphism

$$\mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$$

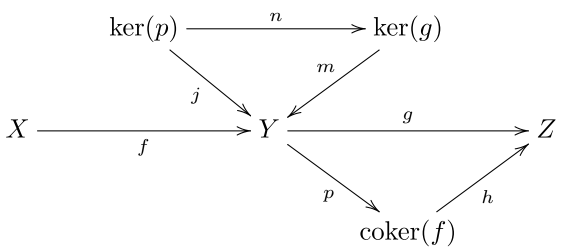
to be an isomorphism, for every morphism f in $\mathcal{C}$. Every module category of a ring is an abelian category, and, as mentioned before, the last condition in the definition is the abstract version of the isomorphism theorem $\mathrm{Im}(f) \cong X/\ker(f)$ in a module category. Other examples of abelian categories include categories of sheaves on topological spaces. The Freyd-Mitchell embedding theorem states that every small abelian category is equivalent to a full subcategory of a module category of some ring A. The notion of exactness can be generalised as follows. A sequence of two composable A-homomorphisms in the category of A-modules

$$U \xrightarrow{\varphi} V \xrightarrow{\psi} W$$

is *exact* if $\mathrm{Im}(\varphi) = \ker(\psi)$. With the technique from above, describing $\mathrm{Im}(\varphi)$ as the kernel of a cokernel of φ, consider a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in an abelian category $\mathcal{C}$, such that $g \circ f = 0$. Let $p : Y \to \mathrm{coker}(f)$ be a cokernel of f. Since $g \circ f = 0$, there is a unique morphism $h : \mathrm{coker}(f) \to Z$ such that $h \circ p = g$. Let $j : \ker(p) \to Y$ be a kernel of p. Thus $p \circ j = 0$, hence $g \circ j = h \circ p \circ j = 0$. Let $m : \ker(g) \to Y$ be a kernel of g. Thus there is a unique morphism $n : \ker(p) \to \ker(q)$ satisfying $j = m \circ n$. We say that the above sequence is *exact* if n is an isomorphism in $\mathcal{C}$.



6.6 Adjoint functors

Definition 6.6.1. Let $\mathcal{C}$, $\mathcal{D}$ be categories and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be covariant functors. We say that $\mathcal{G}$ is *left adjoint to* $\mathcal{F}$ and that $\mathcal{F}$ is *right adjoint to* $\mathcal{G}$, if there is an isomorphism of bifunctors $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$. If $\mathcal{G}$ is left and right adjoint to $\mathcal{F}$ we say that $\mathcal{F}$ and $\mathcal{G}$ are *biadjoint*.

An isomorphism of bifunctors as in Definition 6.6.1 is a family of isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V), U) \cong \mathrm{Hom}_{\mathcal{D}}(V, \mathcal{F}(U)) \ ,$$

with U an object in $\mathcal{C}$ and V an object in $\mathcal{D}$, such that for fixed U we get an isomorphism of contravariant functors $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), U) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(U))$, and for fixed V we get an isomorphism of covariant functors $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V), -) \cong \mathrm{Hom}_{\mathcal{D}}(V, \mathcal{F}(-))$. Such an isomorphism of bifunctors, if it exists, need not be unique. If $\mathcal{C}$, $\mathcal{D}$ are k-linear categories for some commutative ring k, we will always require such an isomorphism of bifunctors to be k-linear. Given an adjunction isomorphism $\Phi : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$, evaluating Φ at an object V in $\mathcal{D}$ and $\mathcal{G}(V)$ yields an isomorphism $\mathrm{Hom}_{\mathcal{D}}(V, \mathcal{F}(\mathcal{G}(V))) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V), \mathcal{G}(V))$. We denote by $f(V) : V \to \mathcal{F}(\mathcal{G}(V))$ the morphism corresponding to $\mathrm{Id}_{\mathcal{G}(V)}$ through this isomorphism; that is, $f(V) = \Phi(V, \mathcal{G}(V))(\mathrm{Id}_{\mathcal{G}(V)})$. One checks that the family of morphisms $f(V)$ defined in this way is a natural transformation

$$f : \mathrm{Id}_{\mathcal{D}} \to \mathcal{F} \circ \mathcal{G}$$

called the *unit* of the adjunction isomorphism Φ, where $\mathrm{Id}_{\mathcal{D}}$ denotes the identity functor on $\mathcal{D}$ (sending every object and every morphism in $\mathcal{D}$ to itself). Similarly, evaluating Φ at an object U in $\mathcal{C}$ and at $\mathcal{F}(U)$ yields an isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(\mathcal{F}(U)), U) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(U), \mathcal{F}(U))$. We denote by $g(U) : \mathcal{G}(\mathcal{F}(U)) \to U$ the morphism corresponding to $\mathrm{Id}_{\mathcal{F}(U)}$ through the isomorphism

$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(\mathcal{F}(U)), U) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(U), \mathcal{F}(U))$; that is, $g(U) = \Phi(\mathcal{F}(U), U)^{-1}(\mathrm{Id}_{\mathcal{F}(U)})$. Again, this is a natural transformation

$$g : \mathcal{G} \circ \mathcal{F} \to \mathrm{Id}_{\mathcal{C}}$$

called the *counit* of the adjunction isomorphism Φ.

An adjunction isomorphism is uniquely determined by its unit and counit. To state this properly we need the following notation. Given two functors $\mathcal{F}$, $\mathcal{F}' : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\varphi : \mathcal{F} \to \mathcal{F}'$, we denote for any functor $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ by $\mathcal{G}\varphi : \mathcal{G}\circ\mathcal{F} \to \mathcal{G}\circ\mathcal{F}'$ the natural transformation given by $(\mathcal{G}\varphi)(U) = \mathcal{G}(\varphi(U)) : \mathcal{G}(\mathcal{F}(U)) \to \mathcal{G}(\mathcal{F}'(U))$ for any object U in $\mathcal{C}$. Similarly, for any functor $\mathcal{H} : \mathcal{E} \to \mathcal{C}$ we denote by $\varphi\mathcal{H} : \mathcal{F} \circ \mathcal{H} \to \mathcal{F}' \circ \mathcal{H}$ the natural transformation given by $\varphi(\mathcal{H}(W)) : \mathcal{F}(\mathcal{H}(W)) \to \mathcal{F}'(\mathcal{H}(W))$ for any object W in $\mathcal{E}$. We denote by $\mathrm{Id}_{\mathcal{F}}$ the identity natural transformation on $\mathcal{F}$, given by the family of identity morphisms $\mathrm{Id}_{\mathcal{F}(U)}$, with U running over the objects of $\mathcal{C}$.

Theorem 6.6.2. *Let $\mathcal{C}$, $\mathcal{D}$ be categories and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be covariant functors.*

(i) Suppose there is an adjunction isomorphism $\Phi : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$. The unit f and counit g of Φ satisfy $(\mathcal{F}g) \circ (f\mathcal{F}) = \mathrm{Id}_{\mathcal{F}}$ and $(g\mathcal{G}) \circ (\mathcal{G}f) = \mathrm{Id}_{\mathcal{G}}$.

(ii) Let $f : \mathrm{Id}_{\mathcal{D}} \to \mathcal{F}\circ\mathcal{G}$ and $g : \mathcal{G}\circ\mathcal{F} \to \mathrm{Id}_{\mathcal{C}}$ be two natural transformations satisfying $(\mathcal{F}g)\circ(f\mathcal{F}) = \mathrm{Id}_{\mathcal{F}}$ and $(g\mathcal{G}) \circ (\mathcal{G}f) = \mathrm{Id}_{\mathcal{G}}$. There is a unique adjunction isomorphism $\Phi : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$ such that f is the unit of Φ and g is the counit of Φ.

(iii) Let $\Phi : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$ be an adjunction isomorphism with unit f and counit g. Then $\Phi(V, U)(\varphi) = \mathcal{F}(\varphi) \circ f(V)$ for any object U in $\mathcal{C}$, any object V in $\mathcal{D}$ and any morphism $\varphi : \mathcal{G}(V) \to U$ in $\mathcal{C}$, and $\Phi(V, U)^{-1}(\psi) = g(U) \circ \mathcal{G}(\psi)$ for any morphism $\psi : V \to \mathcal{F}(U)$ in $\mathcal{D}$. In particular, we have $\varphi = g(U) \circ \mathcal{G}(\mathcal{F}(\varphi) \circ f(V))$ and $\psi = \mathcal{F}(g(U) \circ \mathcal{G}(\psi)) \circ f(V)$.

See e. g. [7, Chapter 2, Section 3] for a proof and more details. The following adjunction is known as the Tensor-Hom adjunction.

Theorem 6.6.3. *Let A, B be k-algebras and let M be an A-B-bimodule. For any A-module U and any B-module V we have natural inverse isomorphisms of k-modules*

$$\begin{cases} \mathrm{Hom}_A(M \otimes_B V, U) & \cong & \mathrm{Hom}_B(V, \mathrm{Hom}_A(M, U)) \\ \varphi & \to & (v \mapsto (m \mapsto \varphi(m \otimes v))) \\ (m \otimes v \mapsto \psi(v)(m)) & \longleftarrow & \psi \end{cases}$$

In particular, the functor $M\otimes_B - : \mathrm{Mod}(B) \to \mathrm{Mod}(A)$ is left adjoint to the functor $\mathrm{Hom}_A(M, -) : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$.

The proof of Theorem 6.6.3 is a straightforward verification.

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