LTCC Differential Geometry & Mathematical Physics

WEEK 3 (with Andy Hone)

Lagrangian mechanics

Configuration space Q

Lagrangian $L : TQ \longrightarrow \mathbb{R}$

Local coords (q, v) on TQ:

write $L = L(q, v)$.

Principle of least action:

Action $S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$

Require $\delta S = 0$.

$\delta S = 0 \Rightarrow$ Euler-Lagrange eq^{n}s.

$q = (q^1, q^2, \ldots, q^n)$ dim $Q = n$

E-L eq^{n}s in components are

$$\frac{\partial L}{\partial q^j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) = 0, \; j = 1, \ldots, n.$$

Examples:

(1) Natural Lagrangian $L = T - V$

$= K.E. - P.E.$

$$L = \frac{1}{2}\sum_{j=1}^{n} m_j \dot{q}_j^2 - V(q)$$

E-L eq^{n}s: $\dfrac{\partial L}{\partial \dot{q}_j} = m_j \dot{q}_j$ (momentum)

$$\left.\begin{aligned} & \\ &\frac{d}{dt}(m_j \dot{q}_j) = \frac{\partial L}{\partial q_j} = -\frac{\partial V}{\partial q_j} \quad (\text{force}) \end{aligned}\right\} \text{Newton II}$$

(2) Riemannian mfd Q with metric g:

$$L = \frac{1}{2} g_{jk}(q)\, \dot{q}^j \dot{q}^k \quad (\textstyle\sum \text{ convention})$$

<u>Exercise:</u> Calculate Euler-Lagrange equations to find

$$\ddot{q}^j + \underbrace{\Gamma^j_{kl}}_{\text{Christoffel symbols}} \dot{q}^k \dot{q}^l = 0,$$

i.e. the geodesic equations for the metric g. Can get same equations by taking action $\int_{t_1}^{t_2} ds = \int_{t_1}^{t_2} \sqrt{g(\dot{q}, \dot{q})}\, dt$

i.e. $L(q, \dot{q}) = \sqrt{g_{jk} \dot{q}_j \dot{q}_k}$

In the pseudo-Riemannian case (e.g. General Relativity),
$L = \frac{1}{2} g_{jk} \dot{q}_j \dot{q}_k$ is the Lagrangian for a test particle in spacetime.

Hamiltonian mechanics (canonical case)

Legendre transformation: map from $TQ \longrightarrow T^*Q$.
In local coords, given vector $v^j \frac{\partial}{\partial q^j}$, get covector $p_j \, dq^j$ given by

$$p_j = \frac{\partial L}{\partial v^j} \rightsquigarrow$$

Poincaré 1-form
$\alpha = p_j \, dq^j$

Hamiltonian function:

$$H(q,p) = \langle p, v \rangle - L(q,v)$$

Want $H : T^*Q \longrightarrow \mathbb{R}$.

To invert Legendre transf., require

$$\left(\frac{\partial^2 L}{\partial v^j \partial v^k} \right) > 0, \text{ so implicit fn thm.}$$

gives $v = v(q,p)$.

Exercise: Euler-Lagrange eq'ns for L + Legendre transf'n $\Rightarrow$ Hamilton's equations for H.

Examples:

① $p_j = m_j \dot{q}^j$ = jth momentum var.

for natural Lagrangian K.E. − P.E.

$\longrightarrow H = $ K.E. + P.E.

2. $L = \frac{1}{2} g_{jk} \dot{q}^j \dot{q}^k \Rightarrow p_j = g_{jk} \dot{q}^k$

(Inverse Legendre) $\Rightarrow \dot{q}^k = g^{kl} p_l$

(where $g^{kl} g_{lj} = \delta^k_j$)

$\Rightarrow H = \frac{1}{2} g^{jk} p_j p_k$.

Basic fact: Hamiltonian H is a conserved quantity of the equations of motion.

So $\dot{q}^j = \frac{\partial H}{\partial p_j}$ & $\dot{p}_j = -\frac{\partial H}{\partial q^j}$, $j = 1, \ldots, n$

$$\Rightarrow \dot{H} = \frac{dH}{dt} = 0$$

$\therefore H = E$ (constant) along trajectories

E = total energy

Hamilton's equations can also be written in terms of canonical Poisson brackets:

$$\dot{q}^j = \{q^j, H\}, \quad \dot{p}_j = \{p_j, H\},$$

$$j = 1, \ldots, n.$$

For any function F on phase space,

$$\frac{dF}{dt} = \{F, H\}$$

so e.g. $\frac{dH}{dt} = \{H, H\} = 0.$

Defn: A Hamiltonian system is completely integrable if $\exists$ n independent functions in involution w.r.t. the Poisson bracket $\{\ ,\ \}$: $H_1 = H, H_2, \ldots, H_n$ with $\{H_j, H_k\} = 0$ $\forall j, k$, on a symplectic manifold of dimension $2n$.

Thm (Liouville) If the system defined by H is completely integrable, then Hamilton's can be reduced to quadratures. The compact connected level sets of $H_1, \ldots, H_n$ are diffeomorphic to tori T^n on which motion is quasiperiodic.

Example: ① Kepler problem in 3D

$$H = \frac{1}{2m}|\underline{p}|^2 - \frac{K}{|\underline{q}|}, \quad K>0$$

$$(\underline{q}, \underline{p}) \in T^*\mathbb{R}^3 \cong \mathbb{R}^6.$$

Angular momentum $\underline{L} = \underline{p} \times \underline{q}$

components satisfy so(3) Poisson algebra $\{L_j, L_k\} = \varepsilon_{jkl} L_l$

Check $\{|\underline{L}|^2, H\} = 0 = \{L_3, H\}$

$$\{|L|^2, L_3\} = 0$$

② Calogero-Moser models:

$$H = \sum_{j=1}^{n} \frac{1}{2} p_j^2 + g^2 \sum_{j \neq k} \frac{1}{(q^j - q^k)^2}$$

$$L = \begin{pmatrix} p_1 & & \frac{ig}{(q^j - q^k)} \\ & p_2 & \\ & & \ddots \\ & & & p_n \end{pmatrix}$$

$\operatorname{tr} L = \sum p_j = H_1$

$\frac{1}{2}\operatorname{tr} L^2 = H = H_2$

$H_j = \frac{1}{j}\operatorname{tr} L^j.$

Lax equation: $\exists$ another matrix $\underline{M}$

such that $\frac{d\underline{L}}{dt} = [\underline{M}, \underline{L}]$

Poisson brackets & non-canonical Hamiltonians

T^*Q is symplectic mfd

symplectic form $\omega \longrightarrow$ skew-pairing on tangent vectors

canonical Poisson bracket $\{\ ,\ \} \longrightarrow$ Poisson tensor J, skew-pairing on covectors

$$\text{vector fields} \underset{\substack{J \\ \text{Poisson tensor}}}{\overset{\substack{\text{symplectic form} \\ \omega}}{\rightleftarrows}} \text{1-forms}$$

In general, a Poisson manifold M has a tensor J which maps 1-forms to vector fields. Given any 2 fns on M, Poisson bracket is

$$\{F, G\} = J(dF, dG).$$

It has the following properties:

- Bilinearity.
- Skew-symmetry: $\{F, G\} = -\{G, F\}$
- Jacobi id.: $\{\{F, G\}, H\} + \text{cyclic} = 0$
- Derivation prop.:
  $$\{F, GH\} = \{F, G\}H + G\{F, H\}.$$