LTCC Differential Geometry + Mathematical Physics – Week 4 w/ Andy Hone

Previously: Differential geometry:
Manifold M + extra structure

Examples
- (pseudo-) Riemannian geometry (M, g)

Metric: nondeg. symmetric covariant $(0,2)$ tensor field $\rightarrow$ inner product on vectors $g(v,w)$
& invertible map $\Gamma(TM) \longrightarrow \Gamma(T^*M)$
$v \longmapsto g(\cdot, v)$

- Symplectic geometry (M, ω)
Symplectic form: closed $d\omega=0$ nondeg. antisymmetric covariant $(0,2)$ tensor field $\rightarrow$ inner product $\omega(v,w)$
& invertible map $\Gamma(TM) \longmapsto \Gamma(T^*M)$
$v \longmapsto \omega(\cdot, v)$

Poisson tensor J: inverse of ω
inverse map $\Gamma(T^*M) \longrightarrow \Gamma(TM)$

Darboux Thm: $\longrightarrow$ locally $\exists$ canonical position/momentum coords s.t. $\omega = dp_j \wedge dq^j$

$\longrightarrow$ Poisson bracket $\{q^j, p_k\} = \delta^j_k$

Poisson bracket of 2 functions on M is

$$\{F, G\} = J(dF, dG)$$

J is $\overset{\text{antisymmetric}}{\wedge}$ contravariant $(2,0)$ tensor (bivector field) $J = J^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = \frac{1}{2} J^{\mu\nu} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu}$,

& $J^{\mu\nu} \omega_{\nu\lambda} = \delta^\mu_\lambda$.

<u>Poisson geometry</u> Symplectic case: Poisson tensor is nondegenerate.

Generalize this to Poisson manifold (M, J)

Skew-symmetric $(2,0)$ tensor + some properties:

Poisson bracket $\{F, G\} = J(dF, dG)$ is a map from $C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$

satisfying

1. Bilinear
2. Skew-symmetric
3. Jacobi identity
4. Derivation property for $\{F, \cdot\}$:

$$\{F, GH\} = \{F, G\}H - G\{F, H\} \quad \forall F, G, H \in C^\infty(M)$$

Quantization: Canonical Poisson bracket

$$\{q_j, p_k\} = \delta_{jk} \xrightarrow[\text{quantization}]{\text{canonical}} [Q_j, P_k] = i\hbar\, \delta_{jk}$$

Q_j, P_j : operators on Hilbert space $\mathcal{H}$.

General Poisson manifold (M, J), local coords (x^μ)

Poisson brackets defined by

$$\{x^\mu, x^\nu\} = J^{\mu\nu}$$

Weinstein's theorem: Locally any Poisson bracket has coords $x = (q_j, p_k, c_\ell)$

where $\{q_j, p_k\} = \delta_{jk}$

& $\{q, q\} = \{p, p\} = \{c, \text{anything}\} = 0$.

Casimirs.

Example $M = \mathbb{R}^3$, coords (L_1, L_2, L_3)

& Poisson bracket

$$\{L_j, L_k\} = \varepsilon_{jk\ell} L_\ell$$

i.e. $\{L_1, L_2\} = L_3$ (+ cyclic)

$$J = \begin{pmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & -L_1 & 0 \end{pmatrix} \quad \ker J = \operatorname{Span}\left\{ \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} \right\}$$

$\boxed{\operatorname{rank} J = 2}$ (except at 0) her J is generated by Casimir function

$$C = (L_1)^2 + (L_2)^2 + (L_3)^2$$

$$dC = 2(L_1\, dL_1 + L_2\, dL_2 + L_3\, dL_3)$$

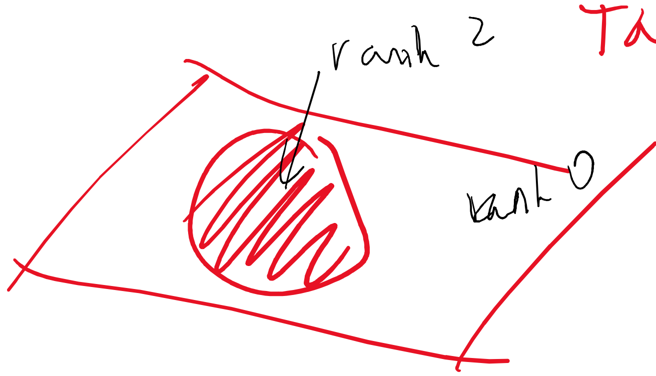
$$J(\cdot, dC) = 0$$

$$\Rightarrow \{F, C\} = 0 \quad \forall\, F \in C^\infty(\mathbb{R}^3).$$

Example: in $\mathbb{R}^2$ coords (x, y)

$$\{x, y\} = f(x,y) \quad (f \text{ arbitrary})$$

Take f to be bump function compactly supported on $x^2+y^2 \leq 1$



$$J = \begin{pmatrix} 0 & f(x,y) \\ -f(x,y) & 0 \end{pmatrix}$$

log-canonical Poisson bracket:

$$\{x, y\} = xy$$

$$\xrightarrow{\text{quantization}} \quad xy = q\, yx \quad \left(\begin{matrix}\text{quantum} \\ \text{plane}\end{matrix}\right)$$

Hamiltonian mechanics on Poisson manifold:

Hamiltonian vector fields

Function $H \longmapsto J(\cdot, dH) = v_H$

In local coords (x^μ), Hamilton's equations are

$$\frac{dx^\mu}{dt} = J^{\mu\nu} \frac{\partial H}{\partial x^\nu}$$

(vector calculus $\dot{\underline{x}} = J \nabla H$)

This form of Hamilton's equations generalizes to field theory / PDEs.

Lagrangian Field Theory (scalar field)

$$S = \int \mathcal{L} \, d^{d+1}x \quad (d+1\text{-dim Minkowski spacetime})$$

Scalar field ϕ, Minkowski metric g

$$\mathcal{L} = \frac{1}{2} g(\partial_\mu \phi, \partial_\mu \phi) - V(\phi)$$
$$= \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi)$$

Least action principle $\delta S = 0$

$\Rightarrow$ Euler-Lagrange eqn's

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\Rightarrow \quad \underline{\partial_\mu \partial^\mu} \phi + \mathcal{V}'(\phi) = 0$$

Wave operator / d'Alembert operator

Examples: ① Free field $\mathcal{V} = \frac{1}{2}\mu^2 \phi^2$

(μ – mass)

Klein-Gordon eqn (linear)

$$\partial_\mu \partial^\mu \phi + \mu^2 \phi = 0$$

② Sine-Gordon $\mathcal{V} = 1 - \cos\phi$

$\to$ sine-Gordon eqn:

$$\partial_\mu \partial^\mu \phi + \sin\phi = 0$$

(1+1-dim: integrable field theory)

Canonical Hamiltonian field theory:

Legendre transformation: momentum density

$$\pi = \frac{\partial \mathcal{L}}{\partial \phi_t}$$

Lagrangian $L = \int \mathcal{L}\, d^d x$ (space integral)

Hamiltonian $H = \int \pi \phi_t \, d^d x - L = \int \left(\frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\phi|^2 + \mathcal{V}\right) d^d x$

Hamilton's equations:

$$\begin{pmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \pi}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \pi} \end{pmatrix}$$

Fréchet derivative $\frac{\delta}{\delta \phi}$ defined by:

$$\frac{d}{d\varepsilon} H[\phi + \varepsilon v] \Big|_{\varepsilon = 0} = \left\langle \frac{\delta H}{\delta \phi}, v \right\rangle \qquad \forall v,$$

where $\langle w, v \rangle$ is L^2 pairing

$$\langle w, v \rangle = \int w v \, d^d x$$

Hamilton's equations for functional

$$H = \int \left(\tfrac{1}{2}\pi^2 + |\nabla \phi|^2 + V(\phi) \right. d^d x$$

are equivalent to Euler-Lagrange eqn's.

This is the basis for canonical quantization, starting from Poisson bracket between fields:

$$\{ \phi(\underline{x}, t), \pi(\underline{y}, t) \} = \delta(\underline{x} - \underline{y})$$

↑ Dirac delta in dimension d.

Hamiltonian PDEs in 1+1 dimensions + integrability

Given a PDE in evolution form

$$u_t = F(u, u_x, u_{xx}, \dots),$$

wish to write it as a Hamiltonian system:

$$u_t = J \frac{\delta H}{\delta u}$$

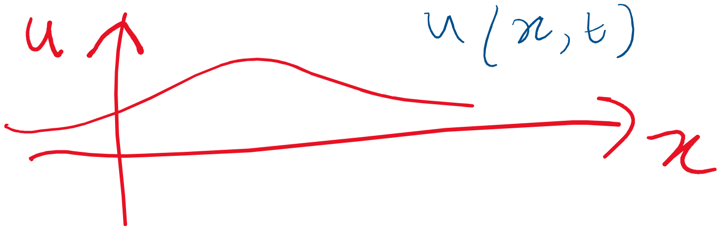
H = functional of u.

J = Hamiltonian operator: skew-symmetric operator, that defines a Poisson bracket on pairs of functionals, according to

$$\{F, G\} = \int \frac{\delta F}{\delta u} J \cdot \frac{\delta G}{\delta u} \, dx.$$

Require $\{\ \}$ is skew-symmetric + satisfies Jacobi identity.

Examples: ① KdV equation (model of shallow water)



$$u_t = u_{xxx} + 6uu_x$$

$$6u\frac{\partial u}{\partial x}$$

$$u_t = \frac{\partial u}{\partial t}, \quad u_{xxx} = \frac{\partial^3 u}{\partial x^3}$$

$$H = \int \left(-\frac{1}{2}u_x^2 + u^3\right) dx$$

$$\frac{\delta H}{\delta u} = \mathcal{E}\circ\left(-\frac{1}{2}u_x^2 + u^3\right)$$

$$= \underbrace{\sum_{j=0}^{\infty} (-\partial_x)^j \frac{\partial}{\partial u_{jx}}}_{\text{Euler operator } \mathcal{E}} \left(-\frac{1}{2}u_x^2 + u^3\right)$$

$$\mathcal{E} = \frac{\partial}{\partial u} - \partial_x\left(\frac{\partial}{\partial u_x}\right) + \partial_x^2\left(\frac{\partial}{\partial u_{xx}}\right) - \dots$$

$J = \partial_x$: skew-adjoint operator + satisfies Jacobi id.

$$\therefore\ u_t = \partial_x\left(u_{xx} + 3u^2\right) = J\frac{\delta H}{\delta u}.$$