Analytical Methods

LTCC Course

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Introduction

This course looks at analytical, rather than computational, methods of tackling difficult problems in applied mathematics. We will primarily apply our attention to differential equations: both ODEs and PDEs.

You might ask, what's the point of searching for analytical solutions or part-solutions when we can solve numerically? There are two killer reasons why analytical techniques remain relevant even in the age of parallel computation:

Programme verification There is always a concern with numerical calculations about whether the code is correct. A helpful check can be to find an extreme case (perhaps setting one parameter to zero, or making it very large or small, or using unphysical but simple boundary conditions) where an analytical solution can be found. If the numerical solution finds the same analytical solution we have increased confidence in its correctness.

Physical insight More importantly, a numerical calculation does not often provide insight into the underlying physics. Sometimes (surprisingly often in practice) the simplified problems presented by taking a limiting case have a simplified physics which nonetheless encapsulates some of the key mechanisms from the full problem – and these mechanisms can then be better understood through analytical methods.

We will be looking at perturbation methods, applied specifically to ODEs and PDEs; many of the principles laid out here will apply equally to other situations such as integral equations (as some of the examples will demonstrate). However, I will intersperse the teaching on perturbation methods (numbered sections) with background techniques in partial differential equations (lettered sections).

References

The principal reference for the perturbation methods is the book by Hinch. In particular, many of the examples and exercises are taken from it. I have also listed three other perturbation methods texts: they are also very good, and your choice here is really a question of style preference.

- Hinch, *Perturbation methods*
- Van Dyke, *Perturbation methods in fluid mechanics*
- Kevorkian & Cole, *Perturbation methods in applied mathematics*
- Bender & Orszag, *Advanced mathematical methods for scientists and engineers*

I am only giving one reference for the background on partial differential equations. I would strongly recommend you to get hold of this book unless your background in PDEs is very strong.

- Weinberger, *A first course in Partial Differential Equations with complex variables and transform methods*

1 Introduction to perturbation methods

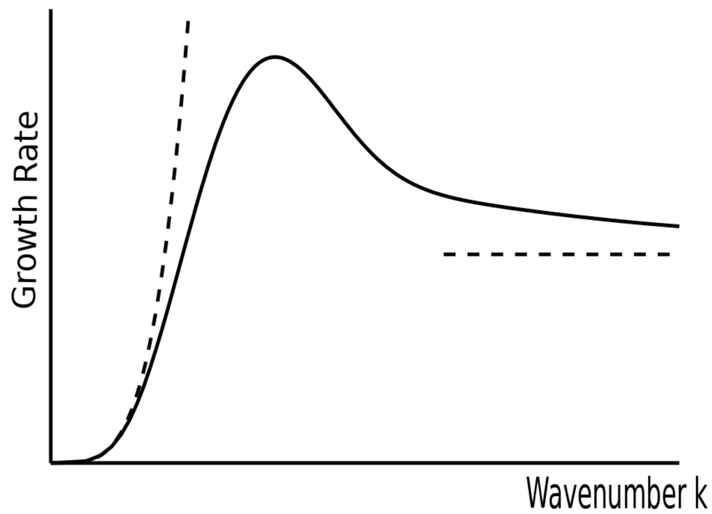
1.1 What are perturbation methods?

Perturbation methods are methods which rely on there being a parameter in the problem that is relatively small: $\varepsilon \ll 1$. The most common example you may have seen before is that of high-Reynolds number fluid mechanics, in which a viscous boundary layer is found close to a solid surface. Note that in this case the standard physical parameter Re is large: our small parameter is $\varepsilon = Re^{-1}$.

1.2 A real research example

This comes from research into polymer flows[1]. I will not present the equations or the working here: but the problem in question is the stability of a polymer extrusion flow. The parameter varied is wavelength: and for both very long waves (wavenumber $k \ll 1$) and very short waves ($k^{-1} \ll 1$) the system is much simplified. The long-wave case, in particular, gives very good insight into the physics of the problem.

If we look at the plot of growth rate of the instability against wavenumber (inverse wavelength):



we can see good agreement between the perturbation method solutions (the dotted lines) and the numerical calculations (solid curve): this kind of agreement gives confidence in the numerics in the middle region where perturbation methods can't help.

[1] H J Wilson & J M Rallison. *J. Non-Newtonian Fluid Mech.*, **72**, 237–251, (1997)

2 Regular perturbation expansions

We're all familiar with the principle of the Taylor expansion: for an analytic function $f(x)$, we can expand close to a point $x = a$ as:

$$f(a+\varepsilon) = f(a) + \varepsilon f'(a) + \tfrac{1}{2}\varepsilon^2 f''(a) + \cdots$$

For general functions $f(x)$ there are many ways this expansion can fail, including lack of convergence of the series, or simply an inability of the series to capture the behaviour of the function; but the paradigm of the expansion in which a small change to x makes a small change to $f(x)$ is a powerful one, and the basis of regular perturbation expansions.

The basic principle and practice of the regular perturbation expansion is:

1. Set $\varepsilon = 0$ and solve the resulting system (solution f_0 for definiteness)
2. Perturb the system by allowing ε to be nonzero (but small in some sense).
3. Formulate the solution to the new, perturbed system as a series

$$f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$

4. Expand the governing equations as a series in ε, collecting terms with equal powers of ε; solve them in turn as far as the solution is required.

2.1 Example differential equation

Suppose we are trying to solve the following differential equation in $x \geq 0$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} + f(x) - \varepsilon f^2(x) = 0, \qquad f(0) = 2. \tag{1}$$

Ignore the fact that we could have solved this equation directly! We'll use it as a model for more complex examples.

We look first at $\varepsilon = 0$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} + f(x) = 0, \qquad f(0) = 2, \qquad \Longrightarrow \qquad f(x) = 2e^{-x}.$$

Now we follow our system and set

$$f = 2e^{-x} + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x) + \cdots$$

where in order to satisfy the initial condition $f(0) = 2$, we will have $f_1(0) = f_2(0) = f_3(0) = \cdots = 0$. Substituting into (1) gives

$$\begin{array}{cccccccc} -2e^{-x} & + & \varepsilon f_1'(x) & + & \varepsilon^2 f_2'(x) & + & \varepsilon^3 f_3'(x) & \\ +2e^{-x} & + & \varepsilon f_1(x) & + & \varepsilon^2 f_2(x) & + & \varepsilon^3 f_3(x) & \\ & - & 4\varepsilon e^{-2x} & - & 4\varepsilon^2 e^{-x} f_1(x) & - & 4\varepsilon^3 e^{-x} f_2(x) & \\ & & & & & - & \varepsilon^3 f_1^2(x) & = \quad O(\varepsilon^4) \end{array}$$

and we can collect powers of ε:

$$\begin{array}{lcrl}
\varepsilon^0 & : & -2e^{-x} + 2e^{-x} & = 0 \\
\varepsilon^1 & : & f_1'(x) + f_1(x) - 4e^{-2x} & = 0 \\
\varepsilon^2 & : & f_2'(x) + f_2(x) - 4e^{-x} f_1(x) & = 0 \\
\varepsilon^3 & : & f_3'(x) + f_3(x) - f_1^2(x) - 4e^{-x} f_2(x) & = 0
\end{array}$$

The order ε^0 (or 1) equation is satisfied automatically. Now we simply solve at each order, applying the boundary conditions as we go along.

Order ε terms.

$$f_1'(x) + f_1(x) = 4e^{-2x} \qquad \Longrightarrow \qquad f_1(x) = -4e^{-2x} + c_1 e^{-x}$$

and the boundary condition $f_1(0) = 0$ gives $c_1 = 4$:

$$f_1(x) = 4(e^{-x} - e^{-2x}).$$

Order ε^2 terms.

The equation becomes

$$f_2'(x) + f_2(x) = 4e^{-x} f_1(x) \implies f_2'(x) + f_2(x) = 16e^{-x}(e^{-x} - e^{-2x})$$

with solution

$$f_2(x) = 8(-2e^{-2x} + e^{-3x}) + c_2 e^{-x}$$

and the boundary condition $f_2(0) = 0$ gives $c_2 = 8$:

$$f_2(x) = 8(e^{-x} - 2e^{-2x} + e^{-3x}).$$

Order ε^3 terms.

The equation is $f_3'(x) + f_3(x) - f_1^2(x) - 4e^{-x} f_2(x) = 0$ which becomes

$$f_3'(x) + f_3(x) = 48(e^{-2x} - 2e^{-3x} + e^{-4x}).$$

The solution to this equation is

$$f_3(x) = 16(-3e^{-2x} + 3e^{-3x} - e^{-4x}) + c_3 e^{-x}.$$

Applying the boundary condition $f_3(0) = 0$ gives $c_3 = 16$ so

$$f_3(x) = 16(e^{-x} - 3e^{-2x} + 3e^{-3x} - e^{-4x}).$$

The solution we have found is:

$$\begin{aligned}
f(x) = 2e^{-x} + 4\varepsilon(e^{-x} - e^{-2x}) + 8\varepsilon^2(e^{-x} - 2e^{-2x} + e^{-3x}) \\
+ 16\varepsilon^3(e^{-x} - 3e^{-2x} + 3e^{-3x} - e^{-4x}) + \cdots
\end{aligned}$$

This is an example of a case where carrying out a perturbation expansion gives us an insight into the full solution. Notice that, for the terms we have calculated,

$$f_n(x) = 2^{n+1} e^{-x}(1 - e^{-x})^n,$$

suggesting a *guessed* full solution

$$f(x) = \sum_{n=0}^{\infty} \varepsilon^n 2^{n+1} e^{-x}(1-e^{-x})^n = 2e^{-x}\sum_{n=0}^{\infty}[2\varepsilon(1-e^{-x})]^n = \frac{2e^{-x}}{1-2\varepsilon(1-e^{-x})}.$$

Having guessed a solution, of course, verifying it is straightforward: this is indeed the correct solution to the ODE of equation (1).

2.2 Example eigenvalue problem

We will find the first-order perturbations of the eigenvalues of the differential equation

$$y'' + \lambda y + \varepsilon y^2 = 0$$

in $0 < x < \pi$, with boundary conditions $y(0) = y(\pi) = 0$.
[Exercise: repeat this with the final term as εy (easy) or εy^3 (harder).]
First we look at the case $\varepsilon = 0$:

$$y'' + \lambda y = 0$$

This has possible solutions:

$$\begin{array}{ll} \lambda < 0 & y = A\cosh[x\sqrt{-\lambda}] + B\sinh[x\sqrt{-\lambda}] \\ \lambda = 0 & y = Ax + B \\ \lambda > 0 & y = A\cos[x\sqrt{\lambda}] + B\sin[x\sqrt{\lambda}] \end{array}$$

The first two solutions can't satisfy both boundary conditions. The third must have $A = 0$ to satisfy the condition $y(0) = 0$, and the second boundary condition leaves us with

$$B\sin[\pi\sqrt{\lambda}] = 0 \qquad \Longrightarrow \qquad \lambda = m^2, \quad m = 1, 2, \ldots$$

Now we return to the full problem, posing regular expansions in both y and λ:

$$y = \sin mx + \varepsilon y_1 + \cdots$$

$$\lambda = m^2 + \varepsilon\lambda_1 + \cdots$$

Substituting in, we obtain for the differential equation:

$$\begin{array}{ccccccc} -m^2\sin mx & + & m^2\sin mx & & & = & 0 \\ \varepsilon y_1'' & + & \varepsilon m^2 y_1 + \varepsilon\lambda_1\sin mx & + & \varepsilon\sin^2 mx & = & 0 \end{array}$$

As we would expect, the order 1 equation is already satisfied, along with the boundary conditions.

Order ε

The ODE at order ε becomes

$$y_1'' + m^2 y_1 = -\lambda_1 \sin mx - \sin^2 mx = -\lambda_1\sin mx + \frac{1}{2}\cos 2mx - \frac{1}{2}.$$

We expect a solution of the form

$$y_1 = A\sin mx + B\cos mx + Cx\cos mx + D\cos 2mx + E$$

and substituting this form back in to the left hand side gives us

$$-2Cm\sin mx - 3m^2D\cos 2mx + Em^2 = -\lambda_1\sin mx + \frac{1}{2}\cos 2mx - \frac{1}{2}$$

which fixes $C = \lambda_1/2m$, $D = -1/6m^2$, $E = -1/2m^2$. The solution is

$$y_1 = A\sin mx + B\cos mx + \frac{\lambda_1}{2m}x\cos mx - \frac{1}{2m^2} - \frac{1}{6m^2}\cos 2mx.$$

Now we apply the boundary conditions to determine the eigenvalue: $y(0) = 0$ gives

$$0 = B - \frac{1}{2m^2} - \frac{1}{6m^2} \qquad B = \frac{2}{3m^2}$$

and then the condition $y(\pi) = 0$ becomes:

$$0 = \frac{2}{3m^2}(-1)^m + \frac{\lambda_1}{2m}\pi(-1)^m - \frac{1}{2m^2} - \frac{1}{6m^2}$$

which simplifies to determine λ_1:

$$\lambda_1 = \frac{4}{3m\pi}[(-1)^m - 1] = \frac{-8}{3m\pi}\begin{cases} 0 & m \text{ even} \\ 1 & m \text{ odd} \end{cases}$$

Thus the eigenvalues become

$$\lambda = 1 - \frac{8\varepsilon}{3\pi}, 4, 9 - \frac{8\varepsilon}{9\pi}, 16, 25 - \frac{8\varepsilon}{15\pi}, \cdots$$

2.3 Warning signs

As I mentioned earlier, the Taylor series model of function behaviour does not always work. The same is true for model systems: and a regular perturbation expansion will not always capture the behaviour of your system. Here are a few of the possible warning signs that things might be going wrong:

One of the powers of ε produces an insoluble equation
By this I don't mean a differential equation with no analytic solution: that is just bad luck. Rather I mean an equation of the form $x_1 + 1 - x_1 = 0$ which cannot be satisfied by any value of x_1.

The equation at $\varepsilon = 0$ doesn't give the right number of solutions
An nth order ODE should have n solutions. If the equation produced by setting $\varepsilon = 0$ has less solutions then this method will not give all the possible solutions to the full equation. This happens when the coefficient of the highest derivative is zero when $\varepsilon = 0$. Equally, for a PDE, if the solution you find at $\varepsilon = 0$ cannot satisfy all your boundary conditions, then a regular expansion will not be enough.

The coefficients of ε can grow without bound
In the case of an expansion $f(x) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \cdots$, the series may not be valid for some values of x if some or all of the $f_i(x)$ become very large. Say, for example, that $f_2(x) \to \infty$ while $f_1(x)$ remains finite. Then $\varepsilon f_1(x)$ is no longer strictly larger than $\varepsilon^2 f_2(x)$ and who knows what even larger terms we may have neglected...

A First-order PDEs

First-order partial differential equations can be tackled with the **method of characteristics**, a powerful tool which also reaches beyond first-order. We'll be looking primarily at equations in two variables, but there is an extension to higher dimensions.

A.1 Wave equation with constant speed

Consider the first-order wave equation with constant speed:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

It responds well to a change of variables:

$$\xi = x + ct \qquad \eta = x - ct$$

The chain rule gives us

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)$$

and so the wave equation is equivalent to

$$2c\frac{\partial u}{\partial \xi} = 0.$$

Integrating gives the general solution $u = F(\eta)$, $u = F(x - ct)$.

But where did we get the change of variables from? The line $x - ct = \text{constant}$ is a line in the x–t plane along which u is constant. This means that if we parametrise this line

$$x = x(r) \qquad t = t(r)$$

then moving along the line by changing r will not change u, i.e.

$$\frac{\mathrm{d}u}{\mathrm{d}r} = 0.$$

This is the underlying principle of the characteristic.

A.2 Variable speed

Let's look now at the variable speed case:

$$\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0.$$

We would like again to find curves along which u is constant. Suppose such a curve is given by $x = x(r)$ and $t = t(r)$. Then, using the chain rule,

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}r} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}r}.$$

We want this to be zero, which is easily achieved if we make this expression the same as the original linear operator:

$$\frac{\mathrm{d}t}{\mathrm{d}r}\frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}r}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0.$$

This gives us the two parametric equations governing the shape of the characteristic curve:

$$\frac{\mathrm{d}t}{\mathrm{d}r} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}r} = c(x,r).$$

These are both ODEs and straightforward to solve.

Example

Look at the equation

$$2\sin\theta\cos 2\phi\frac{\partial u}{\partial \theta} - \frac{\cos\theta\sin 2\phi}{\sin\theta}\frac{\partial u}{\partial \phi} = 0.$$

Suppose that our characteristic is given by $\theta = \theta(r)$, $\phi = \phi(r)$. Then the requirement that u be constant along a characteristic becomes

$$\frac{\partial u}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \frac{\partial u}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}r} = 0.$$

A naïve attempt would be to look at the coupled ODEs

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = 2\sin\theta\cos 2\phi \qquad \frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{\cos\theta\sin 2\phi}{\sin\theta}$$

but we can uncouple them if, before we start, we multiply the original equation by $\sin\theta/\cos\theta\cos 2\phi$:

$$\frac{2\sin^2\theta}{\cos\theta}\frac{\partial u}{\partial \theta} - \frac{\sin 2\phi}{\cos 2\phi}\frac{\partial u}{\partial \phi} = 0,$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{2\sin^2\theta}{\cos\theta} \qquad \frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{\sin 2\phi}{\cos 2\phi}.$$

Now the equations are decoupled, and solving them in turn gives

$$\sin\theta = -\frac{1}{2r} \qquad \sin 2\phi = \exp[C - 2r].$$

Note that we only use a constant of integration in one of these equations; since r is just a parameter, the point $r = 0$ is not defined *a priori*. Effectively, we are making a change of variables from x, t to r, C. We can invert the transformation:

$$C = \ln\sin 2\phi - \frac{1}{\sin\theta} \qquad r = -\frac{1}{2\sin\theta}$$

and since u is constant on this curve, we can deduce the general solution

$$u = F(C) = F\left(\ln\sin 2\phi - \frac{1}{\sin\theta}\right).$$

A.3 More than two dimensions

Now suppose we have the PDE

$$\frac{\partial u}{\partial x} + c_1(x,y,z)\frac{\partial u}{\partial y} + c_2(x,y,z)\frac{\partial u}{\partial z} = 0.$$

Again, we look for a curve on which u is constant; being a curve, it can still be described with a single variable r so we set $x = x(r)$, $y = y(r)$ and $z = z(r)$. Then the chain rule gives

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}r} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}r} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}r}$$

and to make this equal to zero we choose

$$\frac{\mathrm{d}x}{\mathrm{d}r} = 1 \qquad \frac{\mathrm{d}y}{\mathrm{d}r} = c_1(x(r), y(r), z(r)) \qquad \frac{\mathrm{d}z}{\mathrm{d}r} = c_2(x(r), y(r), z(r)).$$

The latter two are now coupled ODEs so we are not guaranteed to be able to find a solution; but sometimes you may be lucky.

Example

Look at the equation

$$\frac{\partial u}{\partial x} + xy\frac{\partial u}{\partial y} - z\ln y\frac{\partial u}{\partial z} = 0.$$

We set $x = x(r)$, $y = y(r)$ and $z = z(r)$ and the chain rule gives

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}r} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}r} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}r}.$$

To match the three coefficients we set:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}r} &= 1 & x(r) &= r \\
\frac{\mathrm{d}y}{\mathrm{d}r} &= xy = ry & y(r) &= y_0 \exp[r^2/2] \\
\frac{\mathrm{d}z}{\mathrm{d}r} &= -z\ln y = -z(\ln y_0 + r^2/2) & z(r) &= z_0 y_0^{-r} \exp[-r^3/6].
\end{aligned}$$

Now we have expressed all points in terms of the three parameters r, y_0 and z_0 and u is independent of r, so the solution is any function of y_0 and z_0. Reversing the change of variables gives

$$r = x \qquad y_0 = y\exp[-x^2/2] \qquad z_0 = zy^r \exp[-r^3/3]$$

and the full solution is

$$u = F(y\exp[-x^2/2]; zy^x \exp[-x^3/3]).$$

A.4 Inhomogeneous case

The characteristic curve is just as fundamental if the equation is not homogeneous, although the function value is no longer constant along characteristics. The method is best seen by example:

$$\frac{\partial u}{\partial t} + 2xt\frac{\partial u}{\partial x} = u$$

over all x, with initial condition (at $t = 0$) $u = x$. We start by finding the characteristic. Here the characteristic is given by

$$\frac{\mathrm{d}t}{\mathrm{d}r} = 1 \qquad t = r \qquad \frac{\mathrm{d}x}{\mathrm{d}r} = 2xr \qquad x = x_0 \exp[r^2].$$

This time our two new variables are r and x_0. Along a specific characteristic we have

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad u = u_0 e^r \text{ or more exactly } u = F(x_0)e^r.$$

We now have a one-parameter family of solutions (parameter x_0): on the curve

$$x = x_0 \exp[t^2], \qquad u = F(x_0)e^t \qquad \text{so} \qquad u = F(x\exp[-t^2])e^t.$$

We need to apply the initial conditions to determine the function F. At $t = 0$ we have $x = x_0$ and $u = F(x_0)$ so the initial condition gives $F(x_0) = x_0$:

$$u = x\exp[-t^2]e^t = x\exp[t - t^2].$$

A.5 Nonlinear homogeneous case

A general first-order homogeneous PDE in two variables can be written as

$$\frac{\partial u}{\partial t} + c(u, x, t)\frac{\partial u}{\partial x} = 0$$

and the method of characteristics still applies (but we expect an implicit solution in general). The characteristic curves are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(u, x, t).$$

Again, we will write the curve parametrically as $t = r$, and x some function of r and a constant x_0 (that is, constant for a given characteristic). Along any characteristic we will have

$$\frac{\mathrm{d}u}{\mathrm{d}r} = 0$$

and so u is constant along a characteristic. We can then make this into a solution everywhere by setting $u = F(x_0)$ on the characteristic specified by x_0.

Since u is constant on our characteristic, the equation of the curve is simply

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(F(x_0), x, t),$$

which is a straightforward ODE in x and t. Once we have solved it we have the characteristic curve

$$t = r \qquad x = G(x_0, F(x_0), r) = G(x_0, u, r).$$

The implicit form of the solution is now:

$$x = G(x_0, u, t) \qquad u = F(x_0)$$

which is a one-parameter family with parameter x_0. In many cases it is possible to rearrange the first equation to obtain x_0 in terms of u and t; then substituting this into the second equation gives the more standard form of the implicit solution.

Example

Consider the advection equation

$$\frac{\partial u}{\partial t} + ux^2 t \frac{\partial u}{\partial x} = 0.$$

Because it is homogeneous, we expect u to be constant along characteristics: so we parameterise with x_0 (constant on each characteristic) and r (which varies along the characteristic) and we can say $u = F(x_0)$.
Now our characteristic curve becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ux^2 t = F(x_0) x^2 t,$$

which we can solve:

$$\int \frac{\mathrm{d}x}{x^2} = F(x_0) \int t \, \mathrm{d}t \qquad -\frac{1}{x} = \frac{1}{2} F(x_0) t^2 - \frac{1}{x_0}$$

$$x = \frac{2x_0}{2 - x_0 F(x_0) t^2}.$$

Thus the characteristic curve and implicit solution are:

$$t = r \qquad x = \frac{2x_0}{2 - x_0 F(x_0) r^2} \qquad u = F(x_0).$$

As above, we can rearrange to get x_0 in terms of x, t and u:

$$x = \frac{2x_0}{2 - x_0 u t^2} \qquad x_0 = \frac{2x}{2 + uxt^2}$$

and the standard implicit form of the solution is

$$u = F\left(\frac{2x}{2 + uxt^2}\right).$$

A.6 Nonlinear inhomogeneous

We are now looking at the most complex of first-order PDEs: those of the form

$$\frac{\partial u}{\partial t} + c(u, x, t)\frac{\partial u}{\partial x} = f(u, x, t)$$

Characteristics still exist in these systems, and they may have important physical properties (for instance, discontinuities in the derivatives of the solution will propagate along them) but unfortunately, since u itself now varies along a characteristic, we can no longer solve even implicitly in general.

Example

Here is a case where we can achieve a little:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \cos t.$$

In this case we can immediately spot one solution

$$u = A + \sin t$$

but can we show that this is not the most general solution?
The characteristics are defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u;$$

let us suppose we know the family of curves $x = f(x_0, r)$, $t = r$. Then we have

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}r} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{\partial u}{\partial t} + \frac{\partial}{\partial r}f(x_0, r)\frac{\partial u}{\partial x}$$

which gives us the two coupled equations

$$\frac{\partial}{\partial r}f(x_0, r) = u \qquad \frac{\mathrm{d}u}{\mathrm{d}r} = \cos r.$$

These are easy to solve in reverse order:

$$u = A(x_0) + \sin r \qquad x = A(x_0)r - \cos r + x_0.$$

Unfortunately, we can't extract x_0 explicitly without choosing the function $A(x_0)$; but note that

$$\begin{array}{lclclcl} A(x_0) & = & \alpha & \Longrightarrow & x_0 = x + \cos t - \alpha t & & u = \alpha + \sin t \\ A(x_0) & = & x_0/\beta & \Longrightarrow & x_0 = \dfrac{\beta\,(x + \cos t)}{(\beta + t)} & & u = \dfrac{x + \cos t}{\beta + t} + \sin t \end{array}$$

so our "spotted" solution is only one of a family of possible solutions.

3 Dimensional Analysis

Dimensional analysis is a very simple and straighforward tool, but with a little flexibility it can be extended to create more general scaling laws: it is always worth considering at the outset of a physical problem.

3.1 Simple concepts

The fundamental units of standard dimensional analysis are the units mass, M; length, L; and time, T. There are others – but we'll start with these. So a mass has dimension M, a velocity dimension LT^{-1}, and a force dimension MLT^{-2}.

The most basic task of dimensional analysis is a check on the sanity of your equations. It is only sensible to add two quantities if they have the same dimensions. Let's look at the Navier–Stokes momentum equation as an example:

$$\rho\left(\partial_t \underline{u} + \underline{u} \cdot \underline{\nabla u}\right) = -\underline{\nabla} p + \eta \nabla^2 \underline{u}.$$

The dimensions of the individual quantities are:

$$[\rho] = ML^{-3} \qquad [\partial_t] = T^{-1} \qquad [\underline{u}] = LT^{-1} \qquad [\nabla] = L^{-1} \qquad [p] = ML^{-1}T^{-2}$$

and let's suppose we don't know what viscosity, η, should be measured in. Then putting all these dimensions into our equation gives:

$$ML^{-3}(LT^{-2}) = ML^{-3}(LT^{-2}) = ML^{-2}T^{-2} = [\eta]L^{-2}LT^{-1}$$

and we can deduce that $[\eta] = ML^{-1}T^{-1}$.

3.2 Extending the concept

Even dimensionless numbers can be part of a dimensions system. For instance, a mole of a substance is defined to be 6.0221367×10^{23} atoms of it. Avogadro's number is just a number – it has no dimensions – and yet in scaling an equation, if one quantity is "per mole" then other quantities added to it must be too. For instance, the ideal gas law

$$pV = nRT$$

has the obvious dimensions

$$[p] = ML^{-1}T^{-2} \qquad [V] = L^3 \qquad [T] = \Theta$$

in which we are using Θ for the dimension of temperature. The other two quantities are less obvious: n is the number of moles of the substance present – just a number, but if we were to define a "dimension" for moles, $\mathcal{M}$, then we could deduce

$$[n] = \mathcal{M} \qquad [R] = ML^2T^{-2}\Theta^{-1}\mathcal{M}^{-1}$$

which does indeed fit: the value of the ideal gas constant R is

$$R = 8.314472\,\mathrm{m}^3\,\mathrm{Pa}\,\mathrm{K}^{-1}\,\mathrm{mol}^{-1} = 8.314472\,\mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-2}\,\mathrm{K}^{-1}\,\mathrm{mol}^{-1}.$$

This concept brings us away from pure dimensional analysis and into the realm of scaling laws.

3.3 Dimensionless parameters

The other critical technique based on dimensional analysis is nondimensionalisation. To quote Andrew Fowler,[2]

> Confronted with, or having created, a mathematical model of a continuous physical system, which consists of a set of differential equations and associated boundary condition, the first thing that an applied mathematician will want to do is non-dimensionalize the system.

Some think this desire is the only real difference between an applied mathematician and a theoretical physicist.

The principle is this: for every dimension relevant in your problem (which may include moles or other such pseudo-dimensions), pick a representative value. It may be that the natural choices are the basic dimensions M, L and T; more often they are not. For the standard example of the Navier–Stokes momentum equations above, we typically choose typical values for the three combinations

$$L \text{ lengthscale} \qquad U = LT^{-1} \text{ velocity} \qquad \eta = ML^{-1}T^{-1} \text{ viscosity.}$$

We introduce new *dimensionless variables* which are just the original variables, scaled with the relevant dimensional combinations. Thus (using a tilde ˜ to denote each dimensionless quantity) we would introduce

$$\underline{u} = U\underline{\tilde{u}} \qquad p = U\eta\tilde{p}/L$$

and scaling lengths and times gives also

$$\tilde{\partial}_t = L\partial_t/U \qquad \tilde{\nabla} = L\nabla$$

which result in the new equation (multiplying by L^2U^{-1}/η):

$$\frac{\rho UL}{\eta}\left(\frac{\partial\underline{\tilde{u}}}{\partial\tilde{t}} + \underline{\tilde{u}}\cdot\underline{\tilde{\nabla}\tilde{u}}\right) = -\underline{\tilde{\nabla}}\tilde{p} + \tilde{\nabla}^2\underline{\tilde{u}}.$$

We have now reduced the number of physical parameters from two (ρ and η) to just one: the Reynolds number $\mathit{Re} = \rho UL/\eta$. It expresses the balance between inertial and viscous terms.

Typically you will have some choice as to which variables to use for scaling and which combinations to use as your dimensionless numbers. Prior work in the field often gives the best clue here: there may be named dimensionless groups and these are often the most convenient choice.

[2] Andrew C. Fowler, *Mathematical Models in the Applied Sciences*, p. 19.

4 Rescaling

After our foray into dimensional analysis at the end of last week, we return to standard perturbation methods now: the next step after we've made our system dimensionless and found that one of the dimensionless parameters is small (or large). In this section we'll look at one of the reasons that our $\varepsilon = 0$ system might not have enough solutions, and introduce a tool that is fundamental to all perturbation systems. We'll start with a very simple example and work up from there.

4.1 Example algebraic equation

Here our model equation is

$$\varepsilon x^2 + x - 1 = 0. \tag{2}$$

Suppose we try a regular perturbation expansion on it. Setting $\varepsilon = 0$ gives

$$x - 1 = 0,$$

with just the one solution $x = 1$. Since we started with a second-degree polynomial we know we have lost one of our solutions; however, if we carry on with the regular perturbation expansion we will get a perfectly valid series for the root near $x = 1$.

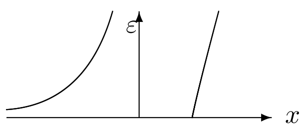
Now let us look at the true solution to see what's gone wrong.

$$x = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}$$

As $\varepsilon \to 0$, the leading-order terms of the two roots are

$$x = 1 + O(\varepsilon); \qquad \text{and} \qquad -\frac{1}{\varepsilon} + O(1).$$

The first of these is amenable to the simplistic approach; we haven't seen the second root because it $\to \infty$ as $\varepsilon \to 0$.



For this second root, let us try a series

$$x = x_{-1}\varepsilon^{-1} + x_0 + \varepsilon x_1 + \cdots$$

We substitute it into (2):

$$\begin{array}{ccccccccc} & x_{-1}^2\varepsilon^{-1} & + & 2x_{-1}x_0 & + & \varepsilon(x_0^2 + 2x_{-1}x_1) & + & \cdots & \\ + & x_{-1}\varepsilon^{-1} & + & x_0 & + & \varepsilon x_1 & + & \cdots & \\ & & - & 1 & & & & & = 0 \end{array}$$

and collecting powers of ε gives:

$$\begin{array}{llrlllrll}
\varepsilon^{-1} & : & x_{-1}^2 + x_{-1} & = 0 & ; & x_{-1} & = & 0 & , & -1 \\
\varepsilon^{0} & : & 2x_{-1}x_0 + x_0 - 1 & = 0 & ; & x_0 & = & 1 & , & -1 \\
\varepsilon^{1} & : & x_0^2 + 2x_{-1}x_1 + x_1 & = 0 & ; & x_1 & = & -1 & , & 1
\end{array}$$

Note that we can now get the expansions for both of the roots using the same method.

4.2 Finding the scaling

What do we do if we can't use the exact solution to tell us about the first term in the series?

We use a trial scaling δ. We put

$$x = \delta(\varepsilon)X$$

with δ being an unknown function of ε, and X being strictly order 1. We call this $X = \mathrm{ord}(1)$: as $\varepsilon \to 0$, X is neither small nor large.

Let's try it for our example equation: $\varepsilon x^2 + x - 1 = 0$. We put in the new form:

$$\varepsilon\delta^2 X^2 + \delta X - 1 = 0$$

and then look at the different possible values of δ. We will only get an order 1 solution for X if the biggest term in the equation is the same size as another term: a **dominant balance** or **distinguished scaling**.

Finding scalings in large systems is more of an art than a science – it's easy to check your scaling works, but finding it in the first place is tricky. But with small systems, it's quite straightforward. I view this process in two ways: one completely systematic (but really only practical with a three-term equation) and the other more of a mental picture.

Systematic method

Since we need the two largest terms to balance, we try all the possible pairs of terms and find the value of δ at which they are the same size. Then for each pair we check that the other term is not bigger than our balancing size.

Balance terms 1 and 2 These two are the same size when $\varepsilon\delta^2 = \delta$ which gives $\delta = \varepsilon^{-1}$. Then both terms 1 and 2 scale as ε^{-1} and term 3 is smaller – so this scaling works.

Balance terms 1 and 3 These two balance when $\varepsilon\delta^2 = 1$ and so $\delta = \varepsilon^{-1/2}$. Then our two terms are both order 1, and term 2 scales as $\varepsilon^{-1/2}$ which is bigger. The balancing terms don't dominate so this scaling is no use.

Balance terms 2 and 3 These two balance when $\delta = 1$, when they are both order 1. Then term 1 is order ε, which is smaller: so we have a working balance at $\delta = 1$.

This process quickly gives us the only two scalings which work: $\delta = \varepsilon^{-1}$ and $\delta = 1$.

Horse-race picture

Think of the terms as horses, which "race" as we change δ. The largest term is considered to be leading, and we are interested in the moment when the lead horse is overtaken: that is, the two biggest terms are equal in size.

The three horses in our case are

$$[\mathbf{A}]\ \ \varepsilon\delta^2 \qquad [\mathbf{B}]\ \ \delta \qquad [\mathbf{C}]\ \ 1$$

and we will start from the point $\delta \approx 0$. Initially, [**C**] is ahead, with [**B**] second and [**A**] a distant third.

As we increase δ, each horse moves according to its power of δ: higher powers move faster (but start further behind). We are looking for the first moment that one of [**A**] or [**B**] catches [**C**]. A quick glance tells us that for [**B**] it will happen at $\delta = 1$ whereas for [**A**] we have to wait until $\delta > 1$. So the first balance is at $\delta = 1$, when [**B**] overtakes [**C**].

Now because [**C**] is the slowest horse (in fact stationary) it will never catch [**B**] again, so we only need to look for the moment (if any) when [**A**] overtakes [**B**]. This is given by $\varepsilon\delta^2 = \delta$ which gives our second balancing scaling of $\delta = \varepsilon^{-1}$.

4.3 Impossible equations: non-integral powers

Try this algebraic equation:

$$(1-\varepsilon)x^2 - 2x + 1 = 0.$$

Setting $\varepsilon = 0$ gives a double root $x = 1$. Now we try an expansion:

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

Substituting in gives

$$\begin{array}{rcccccccc} & 1 & + & 2\varepsilon x_1 & + & \varepsilon^2(x_1^2 + 2x_2) & + & \cdots \\ & & - & \varepsilon & - & 2\varepsilon^2 x_1 & + & \cdots \\ - & 2 & - & 2\varepsilon x_1 & - & 2\varepsilon^2 x_2 & + & \cdots \\ + & 1 & & & & & = & 0 \end{array}$$

At ε^0, as expected, the equation is automatically satisfied. However, at order ε^1, the equation is

$$2x_1 - 1 - 2x_1 = 0 \qquad 1 = 0$$

which we can never satisfy. Something has gone wrong...

In fact in this case we should have expanded in powers of $\varepsilon^{1/2}$. If we set

$$x = 1 + \varepsilon^{1/2}x_{1/2} + \varepsilon x_1 + \cdots$$

then we get

$$\begin{array}{rcccccccc} & 1 & + & 2\varepsilon^{1/2}x_{1/2} & + & \varepsilon(x_{1/2}^2 + 2x_1) & + & \cdots \\ & & & & - & \varepsilon & + & \cdots \\ - & 2 & - & 2\varepsilon^{1/2}x_{1/2} & - & 2\varepsilon x_1 & + & \cdots \\ + & 1 & & & & & = & 0 \end{array}$$

At order ε^0 we are still OK as before; at order $\varepsilon^{1/2}$ we have

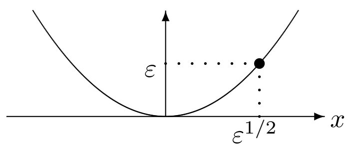
$$2x_{1/2} - 2x_{1/2} = 0$$

which is also automatically satisfied. We don't get to determine anything until we go to order ε^1, where we get

$$x_{1/2}^2 + 2x_1 - 1 - 2x_1 = 0 \qquad x_{1/2}^2 - 1 = 0$$

giving two solutions $x_{1/2} = \pm 1$. Both of these are valid and will lead to valid expansions if we continue.

We could have predicted that there would be trouble when we found the double root: near a quadratic zero of a function, a change of order $\varepsilon^{1/2}$ in x is needed to change the function value by ε:



4.4 Choosing the expansion series

In the example above, if we had begun by defining $\delta = \varepsilon^{1/2}$ we would have had a straightforward regular perturbation series in δ. But how do we go about spotting what series to use?

In practice, it is usually worth trying an obvious series like ε, ε^2, ε^3 or, if there is a distinguished scaling with fractional powers, then a power series based on that. But this trial-and-error method, while quick, is not guaranteed to succeed.

In general, for an equation in x, we can pose a series

$$x \sim x_0\delta_0(\varepsilon) + x_1\delta_1(\varepsilon) + x_2\delta_2(\varepsilon) + \cdots$$

in which x_i is strictly order 1 as $\varepsilon \to 0$ (i.e. tends neither to zero nor infinity) and the series of functions $\delta_i(\varepsilon)$ has $\delta_0(\varepsilon) \gg \delta_1(\varepsilon) \gg \delta_2(\varepsilon) \cdots$ for $\varepsilon \ll 1$.

Then at each order we look for a distinguished scaling. Let us work through an example:

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\varepsilon.$$

In this case there is a solution near $x = 0$, which we will investigate.

First let us sort out the trigonometric term, expanding it as a Taylor series about $x = 0$:

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2}\left[\sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)\right] =$$
$$\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right] = \sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

The governing equation becomes

$$-\frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6) = -\frac{1}{6}\varepsilon.$$

$$x^3 - \frac{x^4}{4} - \frac{x^5}{20} + O(x^6) = \varepsilon.$$

We pose a series

$$x = x_0\delta_0(\varepsilon) + x_1\delta_1(\varepsilon) + \cdots$$

and substitute it. The leading term on the left hand side is $x_0^3\delta_0^3$, and on the right hand side is ε. So we set $\delta_0 = \varepsilon^{1/3}$ and $x_0 = 1$.

Now we have

$$x = \varepsilon^{1/3} + x_1\delta_1(\varepsilon) + \cdots$$

which we substitute into the governing equation. Remembering that $\delta_1 \ll \varepsilon^{1/3}$ and keeping terms up to order $\varepsilon^{2/3}\delta_1$ and $\varepsilon^{4/3}$ (neglecting only terms which are guaranteed to be smaller than one of these), we have

$$3x_1\varepsilon^{2/3}\delta_1 - \varepsilon^{4/3}/4 = 0.$$

To make this work, we need $\delta_1 = \varepsilon^{2/3}$ and then $x_1 = 1/12$.

The first two terms of the solution are:

$$x = \varepsilon^{1/3} + \frac{1}{12}\varepsilon^{2/3} + \cdots$$

4.5 A worse expansion series: Logarithms

Let us consider the equation (with $\varepsilon > 0$):

$$e^{-x} - \varepsilon x = 0.$$

We're looking for the leading-order scaling for x:

$$x \sim x_0\delta_0 + x_1\delta_1 + \cdots$$

As a quick first hack, we need to check we expect a solution at all. Both e^{-x} and $-\varepsilon x$ are decreasing functions so the whole left hand side is a decreasing function of x. At $x = 0$ the function value is 1; for large x, it is negative. Therefore we expect exactly one root, and we expect it to lie in positive x.

In order to see the scaling of the leading term, we will look at the function

$$f(x) = x^{-1}e^{-x} \qquad \text{(we need } f(x) = \varepsilon).$$

It is also a decreasing function, moving from ∞ at $x = 0$ to 0 as $x \to \infty$. We can check the value of $f(x)$ for various values of x, so that we know where to look for the root.

If $x = 1$ then $f(x) = e^{-1}$ which is too large: so we need $x > 1$.

If $x = \varepsilon^{-1}$, then $f(x) = \varepsilon \exp(-\varepsilon^{-1})$ which is exponentially small: so we need $x < \varepsilon^{-1}$.

If $x = \varepsilon^{-\alpha}$ for some fixed positive α, then $f(x) = \varepsilon^{\alpha} \exp(-\varepsilon^{-\alpha})$ which is still exponentially small: so we need a value of x which is larger than 1 but smaller than any negative power of ε. This naturally leads us to the logarithm.

If we try $\delta_0 = \ln(1/\varepsilon)$ (with the inverse present so that δ_0 is positive, which makes everything more intuitive) then the leading-order approximation to $f(x)$ is

$$f(x_0\delta_0) = x_0^{-1}\delta_0^{-1}\exp[-\delta_0 x_0] = \frac{\varepsilon^{x_0}}{x_0 \ln(1/\varepsilon)}$$

Does this work? Let's pick values of x_0 to try.

- If $x_0 = 1$ then $f(x) = \varepsilon / \ln(1/\varepsilon) \ll \varepsilon$.
- If $x_0 = 1/2$ then $f(x) = 2\varepsilon^{1/2} / \ln(1/\varepsilon) \gg \varepsilon$.

These two order-1 values for x_0 bound our root, so we know we have found the right scaling to start with. Once we've got the first scaling it all becomes much easier.

Now let's continue with the series:

$$x = x_0 \ln(1/\varepsilon) + \delta_1 x_1 + \delta_2 x_2 + \cdots$$

Before we go any further, note that $\ln(1/\varepsilon)$ is large and positive, and let us denote

$$L_1 = \ln(1/\varepsilon), \qquad L_2 = \ln\ln(1/\varepsilon) = \ln L_1.$$

The scaling of these terms is $\varepsilon^{-\alpha} \gg L_1 \gg L_2 \gg 1$.

Now on with our expansion. We substitute the first two terms into the governing equation to have:

$$\exp(-[x_0 \ln(1/\varepsilon) + x_1\delta_1 + \cdots]) - \varepsilon[x_0 \ln(1/\varepsilon) + x_1\delta_1 + \cdots] = 0$$

$$\varepsilon^{x_0} \exp(-[x_1\delta_1 + \cdots]) = x_0 \varepsilon \ln(1/\varepsilon) + \cdots .$$

Clearly to make the powers of ε work we need $x_0 = 1$; we then want to fix δ_1 so that

$$\exp(-[x_1\delta_1 + \cdots]) = \ln(1/\varepsilon) + \cdots .$$

For this we need

$$-[x_1\delta_1 + \cdots] = \ln\ln(1/\varepsilon) + \cdots, \qquad x_1\delta_1 = -L_2.$$

We return to the expansion:

$$x = L_1 - L_2 + x_2\delta_2 + \cdots$$

and to the governing equation:

$$\exp(-[L_1 - L_2 + x_2\delta_2 + \cdots]) = \varepsilon[L_1 - L_2 + x_2\delta_2 + \cdots]$$

$$\varepsilon L_1 \exp(-[x_2\delta_2 + \cdots]) = \varepsilon L_1 - \varepsilon L_2 + \cdots$$

$$\exp(-[x_2\delta_2 + \cdots]) = 1 - \frac{L_2}{L_1} + \cdots$$

Now since $L_2 \ll L_1$ we can assume $\delta_2 \ll 1$ and expand the exponential in the usual way:

$$1 - x_2\delta_2 + \cdots = 1 - \frac{L_2}{L_1} + \cdots$$

and so we have found

$$x_2\delta_2 = \frac{L_2}{L_1}.$$

We will carry out just one more term:

$$x = L_1 - L_2 + \frac{L_2}{L_1} + x_3\delta_3 + \cdots$$

$$\exp\left(-[L_1 - L_2 + L_2/L_1 + x_3\delta_3 + \cdots]\right) = \varepsilon[L_1 - L_2 + L_2/L_1 + x_3\delta_3 + \cdots].$$

$$x_3\delta_3 = -L_2/L_1^2 + L_2^2/2L_1^2 + \cdots .$$

In general, if logarithms appear in a problem, only trial and error (as here) or an iterative scheme (see, e.g. Hinch page 12) will give access to a solution. However, solutions are usually expressible in terms of the two logarithmic building blocks L_1 and L_2.

Warning!

Not only are logarithmic expansions horrible to find, they are also a lot less use in practice than the power series we have been looking at. Unless your physical "small parameter" is *extremely* small, L_1 will not be very large and L_2 probably not large at all: so the ordering of terms, while correct in the limit $\varepsilon \to 0$, may not be helpful at a real value of ε. The table below gives an idea of the problem.

ε	L_1	L_2	L_2/L_1	$(L_2^2 - 2L_2)/L_1^2$
10^{-1}	2.303	0.834	0.362	-0.183
10^{-3}	6.908	1.933	0.280	-0.003
10^{-5}	11.51	2.443	0.212	0.008
10^{-7}	16.12	2.780	0.172	0.008
10^{-9}	20.72	3.031	0.146	0.007

B Similarity solutions

Similarity solutions to PDEs are solutions which depend on certain groupings of the independent variables, rather than on each variable separately. The technique for finding them is very similar to scaling analysis in perturbation methods, except that we don't need a small parameter.

I'll show the method by a couple of examples, one linear, the other nonlinear.

B.1 Linear example: the heat equation

The heat equation in one dimension is

$$u_t = \kappa u_{xx}.$$

It is parabolic, which means there is just one family of characteristics. $t =$ constant: but because of the first-order term that fact doesn't help us much.

This form of equation arises often within boundary layers in a PDE: the first-order derivative may be in an unstretched direction and the higher-order derivative come from the component of ∇^2 in a stretched direction, if the coefficient of ∇^2 in the original equation was small (i.e. an advection–diffusion equation with weak diffusion).

We introduce the dilation transformation

$$z = \varepsilon^a x, \qquad s = \varepsilon^b t, \qquad v = \varepsilon^c u$$

under which $\partial_t = \varepsilon^b \partial_s$ and so on, and the PDE becomes

$$\varepsilon^{b-c} v_s = \kappa \varepsilon^{2a-c} v_{zz}.$$

We look for values under which our PDE is unchanged: in this case we have $b - c = 2a - c$ and so $b = 2a$. That tells us that, provided $b = 2a$, if $u(x,t)$ is a solution of the original equation, then so is $\varepsilon^c u(\varepsilon^a x, \varepsilon^b t)$. But what use is this observation?

The key thing is to note that the combinations

$$vs^{-c/b} = \varepsilon^c u(\varepsilon^b t)^{-c/b} = ut^{-c/b} \qquad \text{and} \qquad zs^{-a/b} = \varepsilon^a x(\varepsilon^b t)^{-a/b} = xt^{-a/b}$$

are both unchanged by the transformation , which suggests we look for a solution which combines these two forms:

$$u = t^{c/b} f(xt^{-a/b}).$$

Returning to our specific example, we needed $b = 2a$ which means the combination for the argument of f is $xt^{-1/2} = x/\sqrt{t}$. We introduce a new variable for this combination

$$\xi = x/\sqrt{t} \qquad u = t^{c/b} f(\xi)$$

and substitute into the original equation:

$$u_t = \frac{c}{b} t^{c/b-1} f(\xi) + t^{c/b} f'(\xi) \left(-\frac{1}{2} x t^{-3/2} \right) = \left(\frac{c}{b} f(\xi) - \frac{1}{2} \xi f'(\xi) \right) t^{c/b-1}$$

$$u_{xx} = t^{c/b-1} f''(\xi)$$

$$0 = u_t - \kappa u_{xx} = \left(\frac{c}{b} f(\xi) - \frac{1}{2} \xi f'(\xi) - \kappa f''(\xi) \right) t^{c/b-1}$$

We have reduced a constant-coefficient PDE to a variable-coefficient ODE:

$$\kappa f''(\xi) + \frac{1}{2} \xi f'(\xi) - \frac{c}{b} f(\xi) = 0.$$

For a linear equation like this, the ratio c/b is not determined by the equation and we have some flexibility to use in meeting boundary conditions.

B.1.1 Fixed boundary conditions

Suppose our original equation came with the boundary conditions

$$u(x,0) = 0, \quad x > 0 \qquad u(x,t) \to 0, \quad x \to \infty \qquad u(0,t) = u_0, \quad t > 0.$$

Transforming these into the new variables gives

$$t^{c/b} f(\xi) \to 0, \quad \xi \to \infty, \quad \text{even as } t \to 0 \qquad t^{c/b} f(0) = u_0, \quad t > 0.$$

The first of these gives two conditions: $f(\xi) \to 0$ as $\xi \to \infty$ and also $c/b \geq 0$. The second, on the other hand, can only be satisfied if $c/b = 0$ and then we have the transformation

$$u = f(\xi) \qquad \xi = xt^{-1/2}$$

$$\kappa f''(\xi) + \frac{1}{2}\xi f'(\xi) = 0 \qquad f(\xi) \to 0, \quad \text{as } \xi \to \infty, \qquad f(0) = u_0.$$

We can integrate this once to obtain

$$f'(\xi) = C_1 \exp\left[-\frac{\xi^2}{4\kappa}\right]$$

$$f(\xi) = C_1 \int_0^\xi \exp\left[-\frac{p^2}{4\kappa}\right] \mathrm{d}p + C_2 = C_1(\kappa\pi)^{1/2} \operatorname{erf}\left(\frac{\xi}{2\sqrt{\kappa}}\right) + C_2$$

where $\operatorname{erf}(x) := (2/\sqrt{\pi}) \int_0^x e^{-t^2}\, \mathrm{d}t$. Then the boundary conditions lead to

$$f(\xi) = u_0 \left(1 - \operatorname{erf}\left(\frac{\xi}{2\sqrt{\kappa}}\right)\right) = u_0 \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\kappa}}\right).$$

The solution of the original equation is

$$u = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right).$$

B.1.2 Flux boundary conditions

On the other hand, if we have a flux boundary condition on u:

$$u(x, 0) = 0, \quad x > 0 \qquad u(x, t) \to 0, \quad x \to \infty \qquad u_x(0, t) = Q, \quad t > 0.$$

then we still have the conditions $c/b \geq 0$ and $f(\xi) \to 0$ as $\xi \to \infty$, but now

$$t^{c/b - 1/2} f'(0) = Q, \quad t > 0,$$

which can only be satisfied by taking $c/b = 1/2$. The final transformation is

$$u = t^{1/2} f(\xi) \qquad \xi = x t^{-1/2}$$

giving the ODE and boundary conditions

$$2\kappa f''(\xi) + \xi f'(\xi) - f(\xi) = 0 \qquad f(\xi) \to 0, \quad \xi \to \infty, \qquad f'(0) = Q.$$

It is easy to spot one solution to this equation: $f(\xi) = C_1 \xi$. So we use the reduction-of-order trick and set $f(\xi) = \xi g(\xi)$ to get:

$$2\kappa \xi g''(\xi) + (4\kappa + \xi^2) g'(\xi) = 0$$

Now we can integrate once:

$$g'(\xi) = \frac{C_1}{\xi^2} \exp\left[-\frac{\xi^2}{4\kappa}\right] \qquad g(\xi) = C_1 \int^\xi \frac{1}{p^2} \exp\left[-\frac{p^2}{4\kappa}\right] \mathrm{d}p + C_2$$

$$f(\xi) = C_1 \xi \int^\xi \frac{1}{p^2} \exp\left[-\frac{p^2}{4\kappa}\right] \mathrm{d}p + C_2 \xi.$$

Integrating by parts gives

$$f(\xi) = C_1 \left[-\exp\left[-\frac{\xi^2}{4\kappa}\right] - \frac{\xi\sqrt{\pi}}{2\sqrt{\kappa}} \operatorname{erf}\left(\frac{\xi}{2\sqrt{\kappa}}\right)\right] + C_2 \xi,$$

and after applying the boundary conditions the solution becomes

$$f(\xi) = Q\left(\xi \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\kappa}}\right) - \frac{2\sqrt{\kappa}}{\sqrt{\pi}} \exp\left[-\frac{\xi^2}{4\kappa}\right]\right).$$

B.2 Nonlinear example: KdV equation

The Korteweg–de Vries equation is

$$u_t + 6uu_x + u_{xxx} = 0.$$

Setting $z = \varepsilon^a x$, $s = \varepsilon^b t$ and $v = \varepsilon^c u$ gives

$$\varepsilon^{b-c} v_s + 6\varepsilon^{a-2c} v v_z + \varepsilon^{3a-c} v_{zzz} = 0,$$

which gives us the conditions for invariance:

$$b - c = a - 2c = 3a - c : \qquad b = 3a, \quad c = -2a.$$

The transformation $u = t^{-2/3} f(\xi)$, $\xi = xt^{-1/3}$ converts the KdV equation to

$$t^{-5/3}\left(f'''(\xi) + f'(\xi)\left[6f(\xi) - \frac{\xi}{3}\right] - \frac{2}{3}f(\xi)\right) = 0,$$

$$f'''(\xi) + f'(\xi)\left(6f(\xi) - \frac{\xi}{3}\right) - \frac{2}{3}f(\xi) = 0.$$

This ordinary differential equation can be shown to have the so-called **Painlevé property**, meaning that all movable singular points are poles. A movable singular point is a point where the solution becomes singular but the location of this singularity depends on the arbitrary constants of integration. For instance, the equation $y' = y^2$ has the solution $y = (C - \xi)^{-1}$, which has a singular point whose location depends on the arbitrary constant of integration, C; this equation therefore does have the Painlevé property. The equation $y' = y^3$, on the other hand, does not have this property. There is a conjecture[3], that any ODE obtained as a reduction of a PDE which is solvable by the inverse scattering transform should have the Painlevé property, possibly after a change of variables.

Thus although we can't solve the ODE above in general, the act of deriving it can give us useful information about the original PDE.

C The second-order 1D wave equation

C.1 Homogeneous wave equation with constant speed

The simplest form of the second-order wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Like the first-order wave equation, it responds well to a change of variables:

$$\xi = x + ct \qquad \eta = x - ct$$

which reduces it to

$$-4c^2\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$$

[3] Ablowitz et al., J. Math. Phys. **21**, 715 (1980)

which is solved by

$$u = p(\xi) + q(\eta) = p(x + ct) + q(x - ct)$$

for any differentiable functions p and q. The lines ξ = constant and η = constant are the characteristics, exactly analogous to the characteristics for the first-order equation.

If we add initial conditions

$$u(x, 0) = f(x) \qquad \partial u/\partial t(x, 0) = g(x)$$

then a little algebra gives us **d'Alembert's solution**:

$$u(x, t) = \frac{1}{2}\left[f(x + ct) + f(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y.$$

C.2 Inhomogeneous wave equation

The inhomogeneous wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F(x, t) \tag{3}$$

can be solved in a very similar way. The change of variables results in:

$$-4c^2\frac{\partial^2 u}{\partial\xi\partial\eta} = F\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}\right)$$

which can be integrated directly for any specific function F; however (Weinberger p. 25) it is also possible to carry out the integrals symbolically (paying particular attention to which variable is held constant when integrating with respect to another). The general result is

$$u = p(x + ct) + q(x - ct) + \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x', t')\,\mathrm{d}x'\,\mathrm{d}t'. \tag{4}$$

As this is a linear PDE, the general solution to the inhomogeneous equation is the sum of the general solution to the homogeneous equation (the *CF* in the notation of ODEs) and one particular solution to the full equation (the *PI*). To verify the solution we simply check that the last term of (4) satisfies (3).

Example

Let's consider the example inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 12xt.$$

We can find the solution using the formula (4) but it's not straightforward!

$$u = p(x + ct) + q(x - ct) + \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} 12x't'\,\mathrm{d}x'\,\mathrm{d}t'.$$

Looking just at the integral and carrying out the x' integration first gives

$$\begin{aligned}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} 12x't'\,\mathrm{d}x'\,\mathrm{d}t' &= \int_0^t \left[6(x')^2t'\right]_{x'=x-c(t-t')}^{x+c(t-t')}\,\mathrm{d}t' \\ &= \int_0^t 6\left((x+c(t-t'))^2t' - (x-c(t-t'))^2t'\right)\,\mathrm{d}t' \\ &= \int_0^t 24cxt'(t-t')\,\mathrm{d}t' = \left[cx(12tt'^2 - 8t'^3)\right]_{t'=0}^t \\ &= 4cxt^3,\end{aligned}$$

and the general solution is

$$u = p(x+ct) + q(x-ct) + 2xt^3.$$

For a specific case, though, it is usually more straightforward to work directly from the original equation (with change of variables). In this case when we put $\xi = x+ct$ and $\eta = x-ct$ we obtain

$$-4c^2\frac{\partial^2 u}{\partial\xi\partial\eta} = \frac{3}{c}(\xi+\eta)(\xi-\eta) = \frac{3}{c}(\xi^2-\eta^2).$$

Integrating gives

$$-4c^2\frac{\partial u}{\partial \xi} = p(\xi) + \frac{1}{c}(3\xi^2\eta - \eta^3);$$

$$-4c^2 u = p(\xi) + q(\eta) + \frac{\xi\eta}{c}(\xi+\eta)(\xi-\eta).$$

Converting the coordinates gives

$$\begin{aligned}u &= f(x+ct) + g(x-ct) - \frac{(x+ct)(x-ct)}{4c^3}(2x)(2ct) \\ &= f(x+ct) + g(x-ct) - \frac{xt(x^2-c^2t^2)}{c^2}.\end{aligned}$$

This doesn't immediately look the same; but note that the difference can be absorbed into $f(x+ct)$ and $g(x-ct)$:

$$\begin{aligned}2xt^3 + \frac{xt(x^2-c^2t^2)}{c^2} &= \frac{\left(xt(x^2+c^2t^2)\right)}{c^2} = \frac{(\xi+\eta)(\xi-\eta)((\xi+\eta)^2+(\xi-\eta)^2)}{64c^3} \\ &= \frac{1}{32c^3}\left((\xi^2-\eta^2)(\xi^2+\eta^2)\right) = \frac{1}{32c^3}(\xi^4-\eta^4)\end{aligned}$$

which is the sum of a function of ξ and a function of η.

If you're in any doubt about your solution, plug it back into the original equation: as long as you have the p and q terms, anything that works will be the general solution!

C.3 Varying speed: two sets of characteristics

We saw in the constant-speed case that the characteristic curves were the straight lines

$$x = k_1 + ct \qquad x = k_2 - ct$$

Thus any point (x, t) lies on **two** characteristics, and finding the curves is not quite as straightforward as it was with the first-order wave equation. Characteristics, even for the homogeneous wave equation, are no longer curves along which u is constant.

To understand the wave equation better, let's look at the generalisation to a wavespeed which varies in space:

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\frac{\partial^2 u}{\partial x^2} = 0.$$

The characteristics in this case are curves which satisfy

$$\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 = \frac{1}{c^2(x)}.$$

Not everything carries over from the constant-speed case: the "obvious" change of variables

$$\xi = \int^x \frac{\mathrm{d}x'}{c(x')} + t \qquad \eta = \int^x \frac{\mathrm{d}x'}{c(x')} - t,$$

which makes the characteristics into lines of constant ξ or constant η, only reduces the governing equation to

$$-4\frac{\partial^2 u}{\partial\xi\partial\eta} - c'(x)\left(\frac{\partial u}{\partial\eta} + \frac{\partial u}{\partial\xi}\right) = 0,$$

which has no straightforward solution.

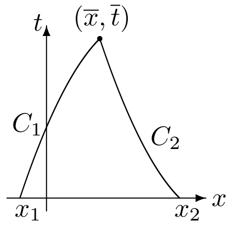
Suppose we specify our initial conditions:

$$u(x, 0) = f(x) \qquad \partial u/\partial t(x, 0) = g(x).$$

This time we will look at the value of the solution at a specific position and time $u(\overline{x}, \overline{t})$. We will prove that the solution depends only on the initial conditions over a range of x determined by the characteristics through our point: so that information propagates along the characteristic curves as in our previous cases.

The characteristics of this problem are curves which satisfy:

$$\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 = \frac{1}{c^2(x)} \qquad \frac{\mathrm{d}t}{\mathrm{d}x} = \pm\frac{1}{c(x)}.$$



We can find the two characteristic curves C_1 and C_2 passing through our point $(\overline{x}, \overline{t})$. These characteristics will reach the initial line $t = 0$ at points x_1 and x_2 respectively (we take $x_1 < x_2$ so that C_1 has positive $\mathrm{d}t/\mathrm{d}x$ and C_2 the negative sign). Either x_1 or x_2 may be infinite.

Now consider two different sets of initial conditions:

$$u_1(x, 0) = f_1(x) \qquad u_2(x, 0) = f_2(x)$$

$$\partial u_1/\partial t(x, 0) = g_1(x) \qquad \partial u_2/\partial t(x, 0) = g_2(x)$$

with $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$ over the range $x_1 \leq x \leq x_2$. If we can show that the corresponding solutions are equal at $(\overline{x}, \overline{t})$ then we know that the function value only depended on the initial conditions between $x_1 < x < x_2$.

The linearity of the problem means that the function $v(x,t) = u_1 - u_2$ satisfies

$$\frac{\partial^2 v}{\partial t^2} - c^2(x)\frac{\partial^2 v}{\partial x^2} = 0 \tag{5}$$

$$v(x,0) = f_1(x) - f_2(x) \qquad \partial v/\partial t(x,0) = g_1(x) - g_2(x)$$

with the initial condition functions both being zero over $x_1 \leq x \leq x_2$.

Multiplying (5) by $(1/c^2(x))(\partial v/\partial t)$ we can rewrite it as

$$\frac{\partial}{\partial t}\left[\frac{1}{2c^2(x)}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right] - \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x}\frac{\partial v}{\partial t}\right] = 0$$

Now we integrate this equation over the region between the line $t = 0$ and the two characteristics C_1 and C_2, meeting at the point $(\overline{x}, \overline{t})$: we integrate the first term first with respect to t and then x, and the second in the other order.

$$\underbrace{\int_{x_1}^{x_2}}_{C_1,\ C_2} \frac{1}{2c^2(x)}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x - \underbrace{\int_0^{\overline{t}}}_{C_2} \frac{\partial v}{\partial x}\frac{\partial v}{\partial t}\,\mathrm{d}t + \underbrace{\int_0^{\overline{t}}}_{C_1} \frac{\partial v}{\partial x}\frac{\partial v}{\partial t}\,\mathrm{d}t = 0$$

The later two can be converted to integrals over x, as we know $\mathrm{d}t/\mathrm{d}x$ on the two characteristics:

$$\underbrace{\int_{x_1}^{x_2}}_{C_1,\ C_2} \frac{1}{2c^2(x)}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial v}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = 0$$

and completing the square gives

$$\underbrace{\int_{x_1}^{x_2}}_{C_1,\ C_2} \left\{\frac{1}{2c^2(x)}\left[\frac{\partial v}{\partial t} + c^2(x)\frac{\partial v}{\partial x}\frac{\mathrm{d}t}{\mathrm{d}x}\right]^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\left[1 - c^2(x)\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right]\right\}\mathrm{d}x = 0$$

Finally, since $(\mathrm{d}t/\mathrm{d}x)^2 = 1/c^2(x)$ on the characteristics, the second term is zero and we have

$$\underbrace{\int_{x_1}^{x_2}}_{C_1,\ C_2} \left\{\frac{1}{2c^2(x)}\left[\frac{\partial v}{\partial t} + c^2(x)\frac{\partial v}{\partial x}\frac{\mathrm{d}t}{\mathrm{d}x}\right]^2\right\}\mathrm{d}x = 0$$

Since the integrand is nonnegative, it must be zero along both characteristics. It follows that

$$\frac{\partial v}{\partial t} + c(x)\frac{\partial v}{\partial x} = 0 \ \text{ on } C_1; \qquad \frac{\partial v}{\partial t} - c(x)\frac{\partial v}{\partial x} = 0 \ \text{ on } C_2.$$

Since $(\overline{x}, \overline{t})$ lies on both characteristics it follows that $\partial v/\partial t = 0$ at $(\overline{x}, \overline{t})$.

Following the same procedure for a point $(\overline{x}, t_0)$ with $t_0 < \overline{t}$, the characteristics will lie within the triangle we used, and will intersect the line $t = 0$ inside the region $x_1 < x < x_2$ where the initial conditions are zero. Thus the working follows identically and we can deduce

$$\frac{\partial v}{\partial t}(\overline{x}, t) = 0 \quad \text{for } 0 \leq t \leq \overline{t}.$$

Since $v(\overline{x}, 0) = 0$ we can integrate wrt t to show $v(\overline{x}, \overline{t}) = 0$.

This completes the proof that $u_1(\overline{x}, \overline{t}) = u_2(\overline{x}, \overline{t})$ and the solution of the wave equation at $(\overline{x}, \overline{t})$ is only affected by information from those initial conditions lying within the characteristics through that point.

D Classification of linear 2nd order PDEs

Consider a linear homogeneous second-order PDE in x and t having constant coefficients and only second-order derivatives:

$$\mathcal{L}[f] = A\frac{\partial^2 f}{\partial t^2} + B\frac{\partial^2 f}{\partial x \partial t} + C\frac{\partial^2 f}{\partial x^2} = 0.$$

We can reduce the differential operator $\mathcal{L}$ to one of three canonical forms using a linear coordinate transformation: if we take

$$\xi = \alpha x + \beta t \qquad \eta = \gamma x + \delta t$$

the operator becomes

$$\begin{aligned}\mathcal{L}[f] = {} & [A\beta^2 + B\alpha\beta + C\alpha^2]\frac{\partial^2 f}{\partial \xi^2} \\ & + [2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma]\frac{\partial^2 f}{\partial \xi \partial \eta} + [A\delta^2 + B\gamma\delta + C\gamma^2]\frac{\partial^2 f}{\partial \eta^2}.\end{aligned}$$

We had great success with the wave equation, choosing coordinates in which only the second term had nonzero coefficient. Can we repeat that in general? We will need to choose α, β, γ and δ so that

$$A\beta^2 + B\alpha\beta + C\alpha^2 = 0 \qquad A\delta^2 + B\gamma\delta + C\gamma^2.$$

Suppose for the sake of the argument that A is nonzero (if it is zero but C is nonzero, switching x and t gives nonzero A; if both A and C are zero we don't need to change variables at all). Then neither α nor γ can be zero and we can divide by them:

$$A\left(\frac{\beta}{\alpha}\right)^2 + B\frac{\beta}{\alpha} + C = 0 \qquad A\left(\frac{\delta}{\gamma}\right)^2 + B\frac{\delta}{\gamma} + C = 0$$

The solutions (identical) to these two constraints are

$$\frac{\beta}{\alpha} = \frac{1}{2A}[-B \pm \sqrt{B^2 - 4AC}] \qquad \frac{\delta}{\gamma} = \frac{1}{2A}[-B \pm \sqrt{B^2 - 4AC}]$$

and the two ratios must be different (for otherwise ξ is a multiple of η): so providing $B^2 - 4AC > 0$ we can choose

$$\alpha = \gamma = 2A \qquad \beta = -B + \sqrt{B^2 - 4AC} \qquad \delta = -B - \sqrt{B^2 - 4AC}$$

$$\xi = 2Ax + [-B + \sqrt{B^2 - 4AC}]t \qquad \eta = 2Ax + [-B - \sqrt{B^2 - 4AC}]t$$

which reduces the whole equation to

$$\mathcal{L}[f] = -4A(B^2 - 4AC)\frac{\partial^2 f}{\partial \xi \partial \eta} = 0,$$

with its general solution

$$\begin{aligned} f &= p(\xi) + q(\eta) \\ f(x,t) &= p(2Ax + [-B + \sqrt{B^2 - 4AC}]t) + q(2Ax + [-B - \sqrt{B^2 - 4AC}]t).\end{aligned}$$

This only works where $B^2 - 4AC > 0$: and in fact this quantity, the discriminant, is very powerful in determining the global behaviour of the PDE solution.

There are three possible cases.

Discriminant positive: Hyperbolic system

This is the case we have just looked at: it can be reduced to a single mixed-derivative term. The example we've seen is the second-order wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

which has $A = 1$, $B = 0$ and $C = -c^2$ giving a discriminant of $4c^2$.

These systems have two families of characteristics along which information propagates, and a typical solution is a combination of travelling waves.

Discriminant zero: Parabolic system

If the discriminant is zero, $B^2 = 4AC$, we can use the transformation

$$\alpha = 2A \qquad \beta = -B \qquad \xi = 2Ax - Bt$$

and any η independent of ξ, and the PDE becomes

$$[A\delta^2 + B\gamma\delta + C\gamma^2]\frac{\partial^2 f}{\partial \eta^2} = 0.$$

The general solution of this equation is (using for illustration $\eta = t$ which is fine as long as $A \neq 0$):

$$\begin{aligned} f &= p(\xi) + \eta q(\xi) \\ f(x,t) &= p(2Ax - Bt) + tq(2Ax - Bt). \end{aligned}$$

A typical parabolic system is the steady one-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

Discriminant negative: Elliptic system

In this case we cannot get rid of the coefficients of $u_{\xi\xi}$ and $u_{\eta\eta}$; however, we can eliminate the mixed derivative using

$$\alpha = 2A \qquad \beta = -B \qquad \gamma = 0 \qquad \delta = \sqrt{4AC - B^2}$$

$$\xi = 2Ax - Bt \qquad \eta = \sqrt{4AC - B^2}t$$

to obtain

$$A(4AC - B^2)\left[\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2}\right] = 0.$$

The classic example of an elliptic PDE is precisely this reduced form: Laplace's equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \nabla^2 u = 0$$

whose discriminant is -1. There are no real characteristics in this case (but we shall see, later, that once we move into complex variable theory, characteristics return to their earlier power).

Typical solutions are energy-minimising surfaces or functions.

D.1 Varying coefficients

If we have a general linear second-order partial differential operator:

$$\mathcal{L}[f] = A(x,t)\frac{\partial^2 f}{\partial t^2} + B(x,t)\frac{\partial^2 f}{\partial x \partial t} + C(x,t)\frac{\partial^2 f}{\partial x^2} + D(x,t)\frac{\partial f}{\partial t} + E(x,t)\frac{\partial f}{\partial x} + F(x,t)f$$

then it can be classified as hyperbolic, parabolic or elliptic at a point (x_0, t_0) according to the value of $B^2(x_0,t_0) - 4A(x_0,t_0)C(x_0,t_0)$, in other words the local discriminant.

D.2 General hyperbolic equation

Suppose that the discriminant is positive everywhere in our domain: then we say the PDE is **hyperbolic in the domain**, and it will have two families of characteristics along which information propagates. We can reduce the second-order terms to the standard form $\partial^2 u/\partial\xi\partial\eta$, although that does not guarantee us a solution: nonetheless, finding the characteristic curves can be very useful.

Instead of using a linear change of variables, we use a general (twice differentiable) transformation:

$$\xi = \xi(x,t) \qquad \eta = \eta(x,t)$$

and the chain rule gives:

$$\begin{aligned}\mathcal{L}[f] = &\left[A\left(\frac{\partial \xi}{\partial t}\right)^2 + B\frac{\partial \xi}{\partial t}\frac{\partial \xi}{\partial x} + C\left(\frac{\partial \xi}{\partial x}\right)^2\right]\frac{\partial^2 f}{\partial \xi^2} \\ &+ \left[2A\frac{\partial \xi}{\partial t}\frac{\partial \eta}{\partial t} + B\frac{\partial \xi}{\partial t}\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial t}\frac{\partial \xi}{\partial x} + 2C\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}\right]\frac{\partial^2 f}{\partial \xi \partial \eta} \\ &+ \left[A\left(\frac{\partial \eta}{\partial t}\right)^2 + B\frac{\partial \eta}{\partial t}\frac{\partial \eta}{\partial x} + C\left(\frac{\partial \eta}{\partial x}\right)^2\right]\frac{\partial^2 f}{\partial \eta^2} \\ &+ [\mathcal{L}[\xi] - F]\frac{\partial f}{\partial \xi} + [\mathcal{L}[\eta] - F]\frac{\partial f}{\partial \eta} + Ff = 0\end{aligned}$$

To zero the unmixed second derivatives, both ratios

$$\frac{\partial\xi/\partial t}{\partial\xi/\partial x} \qquad \frac{\partial\eta/\partial t}{\partial\eta/\partial x}$$

need to be solutions of the quadratic equation

$$Az^2 + Bz + C = 0$$

So now we can calculate the characteristic curves (if we are given A, B and C) by choosing one of these roots for each family.

D.3 Hyperbolic example: finding the characteristics

Since the characteristics only depend on the second-order derivatives, we will just look at the second-order operator for our example (Weinberger p. 45):

$$\mathcal{L}[f] = \frac{\partial^2 f}{\partial t^2} + (5+2x^2)\frac{\partial^2 f}{\partial x \partial t} + (1+x^2)(4+x^2)\frac{\partial^2 f}{\partial x^2}.$$

Note that this is indeed hyperbolic:

$$B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2) = 9.$$

Then we need our two ratios to be

$$\frac{\partial\xi/\partial t}{\partial\xi/\partial x} = \frac{-(5+2x^2)+3}{2} = -1-x^2 \qquad \frac{\partial\eta/\partial t}{\partial\eta/\partial x} = \frac{-(5+2x^2)-3}{2} = -4-x^2.$$

Along the curve $\eta =$ constant,

$$0 = \frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\partial\eta}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\eta}{\partial t} \qquad \text{so} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial\eta/\partial t}{\partial\eta/\partial x} = 4+x^2;$$

similarly, on $\xi =$ constant,

$$0 = \frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\partial\xi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\xi}{\partial t} \qquad \text{so} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial\xi/\partial t}{\partial\xi/\partial x} = 1+x^2.$$

The $\eta =$ constant characteristics are

$$t+\eta = \int \frac{\mathrm{d}x}{4+x^2} = \frac{1}{2}\arctan\frac{x}{2} \qquad x = 2\tan(2t+2\eta)$$

and on $\xi =$ constant we have

$$t+\xi = \int \frac{\mathrm{d}x}{1+x^2} = \arctan x \qquad x = \tan(t+\xi).$$

The change of variables we need is

$$\xi = \arctan x - t \qquad \eta = \frac{1}{2}\arctan\frac{x}{2} - t$$

$$\frac{\partial\xi}{\partial x} = \frac{1}{(1+x^2)} \qquad \mathcal{L}[\xi] = (1+x^2)(4+x^2)\frac{\partial^2\xi}{\partial x^2} = \frac{-2x(4+x^2)}{(1+x^2)}$$

$$\frac{\partial\eta}{\partial x} = \frac{1}{(4+x^2)} \qquad \mathcal{L}[\eta] = (1+x^2)(4+x^2)\frac{\partial^2\eta}{\partial x^2} = \frac{-2x(1+x^2)}{(4+x^2)}$$

and the linear operator becomes

$$\mathcal{L}[f] = -\frac{9}{(1+x^2)(4+x^2)}\frac{\partial^2 f}{\partial\xi\partial\eta} - \frac{2x(4+x^2)}{(1+x^2)}\frac{\partial f}{\partial\xi} - \frac{2x(1+x^2)}{(4+x^2)}\frac{\partial f}{\partial\eta}$$

in which, of course, we should also substitute the new form for x, taken from inverting the function

$$\eta - \xi = \frac{1}{2}\arctan\frac{x}{2} - \arctan x.$$

E Separation of variables: a "lucky" method

Let's look now at the most general constant-coefficient homogeneous linear PDE of second order:

$$A\frac{\partial^2 f}{\partial t^2} + B\frac{\partial^2 f}{\partial x \partial t} + C\frac{\partial^2 f}{\partial x^2} + D\frac{\partial f}{\partial t} + E\frac{\partial f}{\partial x} + Ff = 0.$$

If we can eliminate the mixed-derivative term then we have a chance of using the method of **separation of variables**.

The linear change of variables we were looking at while classifying our equations:

$$\xi = \alpha x + \beta t \qquad \eta = \gamma x + \delta t$$

gave the mixed-derivative term as

$$[2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma]\frac{\partial^2 f}{\partial \xi \partial \eta}$$

It is clear that our four variables are more than enough: we can make a choice under which there is no mixed-derivative term. We'll look later at how to optimise the choice.

E.1 The basics

You will all have seen this method before: I will only run through it briefly. We seek to express our solution as a sum of solutions of the form

$$f(x,t) = X(x)T(t).$$

Substituting this into the governing equation (we've made our change of variables already so there is no mixed derivatives term)

$$A\frac{\partial^2 f}{\partial t^2} + C\frac{\partial^2 f}{\partial x^2} + D\frac{\partial f}{\partial t} + E\frac{\partial f}{\partial x} + Ff = 0$$

gives

$$AX(x)T''(t) + CX''(x)T(t) + DX(x)T'(t) + EX'(x)T(t) + FX(x)T(t) = 0$$

$$\frac{AT''(t)}{T(t)} + \frac{DT'(t)}{T(t)} = -\frac{CX''(x)}{X(x)} - \frac{EX'(x)}{X(x)} - F$$

Now the left hand side of this equation is a function of t only and the right hand side only depends on x, so they must both be a constant, λ, independent of x and t. This insight gives us two ODEs to solve:

$$AT''(t) + DT'(t) - \lambda T(t) = 0 \qquad CX''(x) + EX'(x) + (F + \lambda)X(x) = 0.$$

These give us pairs of solutions, coupled through the value of the constant λ, and typically we write the final solution as

$$f(x,t) = \sum_n X_n(\lambda_n, x)T_n(\lambda_n, t).$$

Example: Laplace in plane polars

Laplace's equation in plane polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}=0 \qquad r^2\frac{\partial^2 f}{\partial r^2}+r\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial\theta^2}=0.$$

The separable solution $f(r,\theta)=R(r)T(\theta)$ gives the coupled ODEs

$$\frac{r^2R''(r)+rR'(r)}{R(r)}=A \qquad \frac{T''(\theta)}{T(\theta)}=-A.$$

We look at the three cases $A>0$, $A<0$ and $A=0$ separately; depending on our domain, some solutions may not be permissible (e.g. if the domain encircles the origin then any solution must be periodic of period 2π in θ).

Positive constant $A=\lambda^2$

$$r^2R''(r)+rR'(r)-\lambda^2R(r)=0 \qquad \text{gives} \qquad R(r)=a_1r^{\lambda}+a_2r^{-\lambda}.$$

$$T''(\theta)=-\lambda^2T(\theta) \qquad \text{gives} \qquad T(\theta)=b_1\cos\lambda\theta+b_2\sin\lambda\theta$$

and the periodicity condition may fix λ to be an integer.

Negative constant $A=-\mu^2$

$$T''(\theta)=\mu^2T(\theta) \qquad \text{gives} \qquad T(\theta)=c_1e^{\mu\theta}+c_2e^{-\mu\theta}$$

and now the periodicity condition cannot be satisfied for $\mu\neq 0$, so these solutions will only be useful in a domain which does not encircle the origin.

$$r^2R''(r)+rR'(r)+\mu^2R(r)=0 \qquad \text{gives} R(r)=c_3\cos\mu\ln r+c_4\sin\mu\ln r.$$

Zero constant $A=0$

$$T''(\theta)=0 \qquad T(\theta)=d_1+d_2\theta \qquad d_2=0.$$

$$r^2R''(r)+rR'(r)=0 \qquad R(r)=d_3+d_4\ln r.$$

The general solution to Laplace's equation in plane polars is then:

$$\begin{aligned}f(r,\theta)=A+B\ln r+\int r^{\lambda}(a(\lambda)\cos[\lambda\theta]+b(\lambda)\sin[\lambda\theta])\,\mathrm{d}\lambda\\ +\int e^{\mu\theta}(c(\mu)\cos[\mu\ln r]+d(\mu)\sin[\mu\ln r])\,\mathrm{d}\mu.\end{aligned} \qquad (6)$$

E.2 Boundary conditions

Of course, Laplace's equation is also separable (has no mixed derivatives) in Cartesian coordinates; and a similar procedure produces the general solution

$$\begin{aligned}f(x,y)&=(\alpha x+\beta)(\gamma y+\delta)\\ &+\int(a(\lambda)\cos\lambda x+b(\lambda)\sin\lambda x)(c(\lambda)e^{\lambda y}+d(\lambda)e^{-\lambda y})\,\mathrm{d}\lambda\\ &+\int(A(\lambda)\cos\lambda y+B(\lambda)\sin\lambda y)(C(\lambda)e^{\lambda x}+D(\lambda)e^{-\lambda x})\,\mathrm{d}\lambda\end{aligned}$$

so how do we know which solution to use?

The simple answer is that the boundary conditions are crucial. Any second order PDE possesses a range of possible coordinates in which it has no mixed derivatives: and the boundary conditions of the specific problem to be solved must inform our choice.

We need the following conditions to be satisfied:

Separable equation
The differential equation must be separable: that is, there are no mixed derivatives and, if the coefficients depend on η and ξ, then (after multiplication of the whole equation by some function if necessary) the derivatives w.r.t. η have coefficients which depend only on η and those w.r.t. ξ have coefficients which depend only on ξ. The coefficient of the no-derivatives term must be at worst the sum of a function of η and a function of ξ.

$$\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{(t+1)^2}\frac{\partial^2 u}{\partial x^2} = 0 \text{ is OK} \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \cos(xt)u = 0 \text{ is not.}$$

Boundary conditions on coordinate lines
All the boundary conditions in the problem must be located along lines η = constant or ξ = constant. This does include the possibility of a boundary condition as one variable $\to \infty$.

Correct type of boundary conditions
Along a line η = constant, the boundary condition must not involve any partial derivatives with respect to ξ; and the coefficients of derivatives involved in the boundary conditions must not vary with ξ.

$$\frac{\partial f}{\partial \eta}(0,\xi) = g(\xi) \text{ is OK} \qquad \left(\frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \xi}\right)(0,\xi) = 0 \text{ is not.}$$

The equivalent condition is required of the boundary conditions along a line ξ = constant.

Realistically, the boundary conditions are likely to completely constrain the coordinates we use if we wish to use separation of variables; and if the coordinates that work for the boundary conditions don't work for the PDE, there's very little we can do about it.

Example

[Weinberger p. 70.]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is a flukey one: it looks like it won't work but a bit of cunning will get us there. First we try the standard separable solution:

$$u = X(x)Y(y) \qquad X''(x)Y(y) + X'(x)Y'(y) + X(x)Y''(y) = 0$$

and then look at Y''/Y:

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} + \frac{X'(x)Y'(y)}{X(x)Y(y)}.$$

Taking the partial derivative of this equation w.r.t. x (and noting that the left hand side is independent of x) gives

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{X''(x)}{X(x)}\right) + \frac{Y'(y)}{Y(y)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{X'(x)}{X(x)}\right) \\ &= \frac{X'''(x)X(x) - X''(x)X'(x)}{X^2(x)} + \frac{Y'(y)}{Y(y)}\left(\frac{X''(x)X(x) - X'(x)^2}{X^2(x)}\right), \end{aligned}$$

which is separable if we divide by the bracketed term on the right:

$$-\frac{X'''(x)X(x) - X''(x)X'(x)}{X''(x)X(x) - X'(x)^2} = \frac{Y'(y)}{Y(y)} = 2\lambda$$

Now we proceed as before: solve

$$Y'(y) = 2\lambda Y(y) \qquad Y(y) = e^{2\lambda y}$$

If we were to carry on with this equation we would have to solve

$$X'''(x)X(x) - X''(x)X'(x) + 2\lambda X''(x)X(x) - 2\lambda X'(x)^2 = 0$$

but now that we know Y, we can return to the original equation:

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} + \frac{X'(x)Y'(y)}{X(x)Y(y)}: \qquad -4\lambda^2 = \frac{X''(x)}{X(x)} + 2\lambda\frac{X'(x)}{X(x)}.$$

$$X(x) = e^{-\lambda x}(a\cos\sqrt{3}\lambda x + b\sin\sqrt{3}\lambda x)$$

and the general solution is

$$u(x,y) = \sum_\lambda \exp\left[\lambda(2y - x)\right](a_\lambda\cos\sqrt{3}\lambda x + b_\lambda\sin\sqrt{3}\lambda x).$$

The moral of this story is: if your boundary conditions look suitable for separation of variables, but your equation doesn't, don't despair – at least not until you've had a go!

5 Scalings with differential equations

5.1 Stretched coordinates

Consider the first-order linear differential equation

$$\varepsilon\frac{\mathrm{d}f}{\mathrm{d}x} + f = 0.$$

Since it is first order, we expect a single solution to the homogeneous equation. If we try our standard method and set $\varepsilon = 0$ we get $f = 0$ which is clearly not a good first term of an expansion!

Solving the differential equation directly gives

$$f = A_0 \exp\left[-x/\varepsilon\right].$$

This gives us the clue that what we **should** have done was change to a *stretched variable* $z = x/\varepsilon$.

Let us ignore the full solution and simply make that substitution in our governing equation. Note that $\mathrm{d}f/\mathrm{d}x = \mathrm{d}f/\mathrm{d}z \ \mathrm{d}z/\mathrm{d}x = \varepsilon^{-1}\mathrm{d}f/\mathrm{d}z$.

$$\varepsilon\varepsilon^{-1}\frac{\mathrm{d}f}{\mathrm{d}z} + f = 0 \qquad \frac{\mathrm{d}f}{\mathrm{d}z} + f = 0.$$

Now the two terms balance: that is, they are the same order in ε. Clearly the solution to this equation is now $A_0 \exp[-z]$ and we have found the result.

This is a general principle. For a polynomial, we look for a distinguished scaling of the quantity we are trying to find. For a differential equation, we look for a stretched version of the independent variable.

The process is very similar to that for a polynomial. We use a trial scaling δ and set

$$x = a + \delta(\varepsilon)X.$$

Then we vary δ, looking for values at which the two largest terms in the scaled equation balance.

Let's work through the process for the following equation:

$$\varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} - f = 0.$$

Again, we note that if $x = a + \delta X$ then $\mathrm{d}/\mathrm{d}x = \mathrm{d}/\mathrm{d}X \ \mathrm{d}X/\mathrm{d}x = \delta^{-1}\mathrm{d}/\mathrm{d}X$. We substitute in these scalings, and then look at gradually increasing δ:

$$[\mathbf{A}] \ \ \varepsilon\delta^{-2} \qquad [\mathbf{B}] \ \ \delta^{-1} \qquad [\mathbf{C}] \ \ 1$$

For small δ term **[A]** is the largest; as δ increases term **[B]** catches up first at $\delta = \varepsilon$. Then **[C]** catches **[B]** at $\delta = 1$ so the two distinguished stretches are $\delta = \varepsilon$ and $\delta = 1$.

For $\delta = 1$ we can treat this as a regular perturbation expansion:

$$f = f_0(x) + \varepsilon f_1(x) + \cdots$$

$$\begin{array}{ccccccc} & & f_0' & - & f_0 & = & 0 \\ \varepsilon f_0'' & + & \varepsilon f_1' & - & f_1 & = & 0 \end{array}$$

At leading order we have

$$f_0' - f_0 = 0 \qquad f_0(x) = a_0 e^x,$$

and the next order becomes

$$f_1' - f_1 = -a_0 e^x \qquad f_1(x) = a_1 e^x - a_0 x e^x$$

so the regular solution begins

$$f(x) \sim a_0 e^x + \varepsilon(a_1 - a_0 x)e^x + \cdots$$

For $\delta = \varepsilon$ we use our new variable $X = \varepsilon^{-1}(x - a)$ and work with the new governing equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}X^2} + \frac{\mathrm{d}f}{\mathrm{d}X} - \varepsilon f = 0$$

Again, with the new scaling, we try a regular perturbation expansion:

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$

We substitute this in and collect powers of ε:

$$\begin{array}{ccccccc} f_{0XX} & + & f_{0X} & & & = & 0 \\ \varepsilon f_{1XX} & + & \varepsilon f_{1X} & - & \varepsilon f_0 & = & 0 \\ \varepsilon^2 f_{2XX} & + & \varepsilon^2 f_{2X} & - & \varepsilon^2 f_1 & = & 0 \end{array}$$

We then solve at each order:

$$\begin{array}{llll} \varepsilon^0 & : & f_{0XX} + f_{0X} = 0 & f_0 = A_0 + B_0 e^{-X} \\ \varepsilon^1 & : & f_{1XX} + f_{1X} - f_0 = 0 & f_1 = A_0 X - B_0 X e^{-X} + A_1 + B_1 e^{-X} \end{array}$$

and so on. Of course, without boundary conditions to apply, this process spawns large numbers of unknown constants. Rescaling to our original variable completes the process:

$$\begin{aligned} f(x) \sim{} & A_0 + B_0 \exp\left[-\frac{(x-a)}{\varepsilon}\right] \\ & + \varepsilon\left\{A_1 + A_0\left(\frac{x-a}{\varepsilon}\right) + \left(B_1 - B_0\left(\frac{x-a}{\varepsilon}\right)\right)\exp\left[-\frac{(x-a)}{\varepsilon}\right]\right\} + \cdots \end{aligned}$$

Note that this expansion is only valid where $X = (x-a)/\varepsilon$ is order 1: that is, for x close to the (unknown) value a.

5.2 Must two terms dominate?

In fact we've been rather harsh in our conditions. To find all roots of a polynomial, we only ever consider scalings where the two largest terms balance. But for a differential equation we can, if we like, be more relaxed. We **must** include at least one scaling in which the highest-order derivative participates, otherwise we have lost one solution of our equation; but it is possible to have a solution in which a derivative (usually the highest derivative) dominates alone. Sometimes this is a (non-fatal) sign that we could have chosen our scaling better; sometimes, in complicated systems, it's unavoidable.

Example

$$\frac{\mathrm{d}f}{\mathrm{d}x} + \varepsilon f = 0 \qquad \text{with boundary condition } f(0) = C.$$

Of course in this case we can either find the scaling instantly ($x \sim \varepsilon^{-1}$) or solve the whole equation. But suppose instead we were to try a regular expansion:

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots$$

$$\begin{array}{ccccccccc} f_0' & + & \varepsilon f_1' & + & \varepsilon^2 f_2' & + & \varepsilon^3 f_3' & + & \cdots \\ & + & \varepsilon f_0 & + & \varepsilon^2 f_1 & + & \varepsilon^3 f_2 & = & 0 \end{array}$$

then solving at each order in turn, applying the boundary condition, gives

$$f_0' = 0 \qquad f_0 = a_0 \qquad f_0 = C$$

$$f_1' + C = 0 \qquad f_1 = a_1 - Cx \qquad f_1 = -Cx$$
$$f_2' - Cx = 0 \qquad f_2 = a_2 + \tfrac{1}{2}Cx^2 \qquad f_2 = \tfrac{1}{2}Cx^2$$

which is a perfectly good regular expansion for the true solution:

$$f = C\left\{1 - \varepsilon x + \tfrac{1}{2}\varepsilon x^2 + \cdots\right\} \qquad f = C\exp(-\varepsilon x).$$

5.3 Nonlinear differential equations: scale and stretch

Recall that for a linear differential equation, if f is a solution then so is Cf for any constant C. So if $f(x;\varepsilon)$ is a solution as an asymptotic expansion, then Cf is a valid asymptotic solution **even if C is an arbitrary function of ε.**

The same is not true of nonlinear differential equations. Suppose we are looking at the equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \varepsilon f(x)\frac{\mathrm{d}f}{\mathrm{d}x} + f^2(x) = 0$$

There are two different types of scaling we can apply: we can scale f, or we can stretch x. To get all valid scalings we need to do both of these at once.

Let us take $f = \varepsilon^\alpha F$ where F is strictly ord(1), and $x = a + \varepsilon^\beta z$ with z also strictly ord(1). Then a derivative scales like $\mathrm{d}/\mathrm{d}x \sim \varepsilon^{-\beta}\mathrm{d}/\mathrm{d}z$ and we can look at the scalings of all our terms:

$$\begin{array}{ccccccc} \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} & + & \varepsilon f(x)\dfrac{\mathrm{d}f}{\mathrm{d}x} & + & f^2(x) & = & 0 \\ \varepsilon^\alpha\varepsilon^{-2\beta} & & \varepsilon\varepsilon^{2\alpha}\varepsilon^{-\beta} & & \varepsilon^{2\alpha} & & \end{array}$$

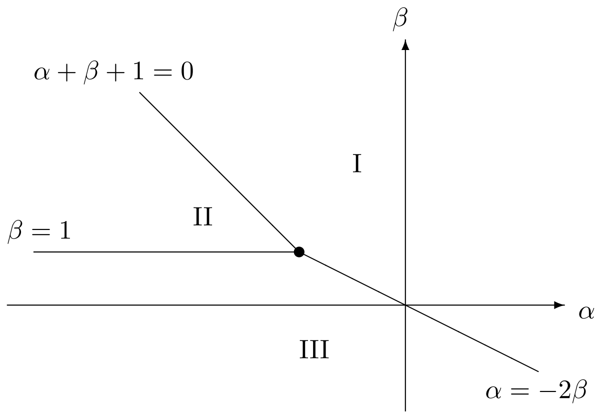
As always with three terms in the equation, there are three possible balances.

- For terms I and II to balance, we need $\alpha - 2\beta = 2\alpha + 1 - \beta$. This gives $\alpha + \beta + 1 = 0$, so that terms I and II scale as $\varepsilon^{2+3\alpha}$, and term III scales as $\varepsilon^{2\alpha}$. We need the balancing terms to dominate, so we also need $2\alpha > 2+3\alpha$ which gives $\alpha < -2$.

- For terms I and III to balance, we need $\alpha - 2\beta = 2\alpha$. This gives $\alpha = -2\beta$, so that terms I and III scale as $\varepsilon^{2\alpha}$ and term II scales as $\varepsilon^{1+5\alpha/2}$. Again, we need the non-balancing term to be smaller than the others, so we need $1 + 5\alpha/2 > 2\alpha$, i.e. $\alpha > -2$.

- Finally, to balance terms II and III, we need $2\alpha - \beta + 1 = 2\alpha$ which gives $\beta = 1$. Then terms II and III scale as $\varepsilon^{2\alpha}$ and term I scales as $\varepsilon^{\alpha-2}$, so to make term I smaller than the others we need $\alpha - 2 > 2\alpha$, giving $\alpha < -2$.

We can plot the lines in the α–β plane where these balances occur, and in the regions between, which term (I, II or III) dominates:



We can see that there is a distinguished scaling $\alpha = -2$, $\beta = 1$ where all three terms balance. If we apply this scaling to have $z = (x - a)/\varepsilon$ and $F = \varepsilon^2 f$ then the governing ODE for $F(z)$ (after multiplication of the whole equation by ε^4) becomes

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + F\frac{\mathrm{d}F}{\mathrm{d}z} + F^2 = 0.$$

This is very nice: but it may not always be appropriate: the boundary conditions may fix the size of either f or x, in which case the best you can do may be one of the simple balance points (i.e. a point (α, β) lying on one of the lines in the diagram).

5.4 Scale and stretch with linear differential equations

Scaling might not seem useful in a linear equation: but if the equation is expressed in terms of more than one physical variable, the relative scales of the different variables are not necessarily obvious beforehand. To give an example I'm using the ODEs which result from a particular linear stability problem I've studied[4]: I've thrown away a few terms to make it less daunting, but there's still plenty to worry about!

There are 7 variables: a streamfunction ψ, three stress components s_1, s_2 and s_3, the pressure p, and two polymer stresses t_1 and t_2.

There are also two physical parameters: k (a wavenumber) and l (a diffusion lengthscale). Either of them can be small.

$$ks_1 + s_2' = 0 \qquad ks_2 + s_3' = 0$$

$$\begin{aligned} s_1 &= -p + 2k\psi' + t_1 \\ s_2 &= \psi'' + t_2 \\ s_3 &= -p - 2k\psi' - t_1 \\ t_1 - l^2 t_1'' &= 2k\psi' + 2(\psi'' + k^2\psi) \\ t_2 - l^2 t_2'' &= \psi'' - k^2\psi \end{aligned}$$

[4] H J Wilson & S M Fielding. *J. Non-Newtonian Fluid Mech.*, **138**, 181–196, (2006)

Small k: regular expansion

Physical understanding allows us to predict that the expansion for small k will be regular: this is because small k means we're studying long waves, and we don't expect anything to happen on a very short lengthscale for long waves.

Since the whole system is linear, there's no amplitude, so we can freely choose one variable to make strictly order 1. Here we'll choose the streamfunction, ψ:

$$\psi \sim \psi_0 + k\psi_1 + \cdots$$

Now looking at the last two equations, and assuming that t_1 and t_2 are the same size, the dominant terms on their right hand sides are ψ'' in both cases: so we can take t_1 and t_2 to be order 1 as well.

The tricky part comes in deciding the size of the s_i terms and p. They *could* all be order 1; then the first two equations would give us, at leading order,

$$s_2' = s_3' = 0;$$

in fact this is a "second-best" scaling and we can do better by allowing s_1, s_3 and p to have singular scalings:

$$s_1 = k^{-1}\overline{s}_1 + O(1) \qquad s_3 = k^{-1}\overline{s}_3 + O(1) \qquad p = k^{-1}\overline{p} + O(1).$$

Then our set of ODEs at leading order is

$$\overline{s}_1 + s_2' = 0 \qquad \overline{s}_3' = 0 \qquad \overline{s}_1 = \overline{s}_3 = -\overline{p}$$

$$\begin{aligned} s_{2,0} &= \psi_0'' + t_2 \\ t_1 - l^2 t_1'' &= 2\psi_0'' \\ t_2 - l^2 t_2'' &= \psi_0'' \end{aligned}$$

Small l: regular expansion

When the lengthscale l is small, there are two expansions. The first is regular and in fact does not need any scaling at all: the leading order equations are simply

$$ks_1 + s_2' = 0 \qquad ks_2 + s_3' = 0$$

$$\begin{aligned} s_1 &= -p + 2k\psi' + t_1 \\ s_2 &= \psi'' + t_2 \\ s_3 &= -p - 2k\psi' - t_1 \\ t_1 &= 2k\psi' + 2(\psi'' + k^2\psi) \\ t_2 &= \psi'' - k^2\psi \end{aligned}$$

However, we have thrown away our highest-order derivatives of t_1 and t_2 in making this expansion, so we know there must be a singular expansion as well.

Small l: stretching coordinate

Because our equations are linear, t_1 will always be larger than $l^2 t_1''$ no matter how we scale t_1; in order to bring back the highest derivatives we will have to stretch the underlying coordinate.

The terms we are concerned about are $l^2 t_1''$ and $l^2 t_2''$, and they appear in equations with terms t_1 and t_2. Balancing these two types of term immediately suggests a stretch $x = a + lz$ (where x is our original coordinate). Applying this to the original equations, and using prime now to represent derivatives w.r.t. z, we have

$$ks_1 + l^{-1}s_2' = 0 \qquad ks_2 + l^{-1}s_3' = 0$$

$$\begin{aligned}
s_1 &= -p + 2kl^{-1}\psi' + t_1 \\
s_2 &= l^{-2}\psi'' + t_2 \\
s_3 &= -p - 2kl^{-1}\psi' - t_1 \\
t_1 - t_1'' &= 2kl^{-1}\psi' + 2(l^{-2}\psi'' + k^2\psi) \\
t_2 - t_2'' &= l^{-2}\psi'' - k^2\psi
\end{aligned}$$

Again, we will start by fixing ψ strictly order 1: then it appears from the last two equations that t_1 and t_2 will be order l^{-2} and (following through) so will all the stresses s_i. The leading-order equations (in the new scaled variables) in this case are:

$$s_2' = s_3' = 0 \qquad t_1 - t_1'' = 2\psi'' \qquad t_2 - t_2'' = \psi''$$

$$s_1 = -p + t_1 \qquad s_2 = \psi'' + t_2 \qquad s_3 = -p - t_1$$

But that's not the only scaling that works.

If we continue with ψ being strictly order 1, but consider the possibility that its leading-order term is a constant, then the forcing terms in the t equations are order 1, and we can use the same trick as for the small-k case to get s_1 involved in the first equation: put p order l^{-1}, then s_1 and s_3 are also order l^{-1} and the leading-order equations are:

$$s_1 = s_3 = -p \qquad s_2 = l^{-2}\psi'' + t_2$$

$$ks_1 + s_2' = s_3' = 0 \qquad t_1 - t_1'' = 2k^2\psi \qquad t_2 - t_2'' = -k^2\psi$$

This seems less obvious and perhaps even less convincing than the straighforward scaling above: but in the real problem I was solving, this scaling gave the balances we needed.

6 Matching: Boundary Layers

Consider the following equation (rather similar to the example we used in section 5.1):

$$\varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}+\frac{\mathrm{d}f}{\mathrm{d}x}+f=0$$

There are two solutions. One is regular:

$$f=f_0(x)+\varepsilon f_1(x)+\cdots$$

Substituting gives, at order 1,

$$f_0'+f_0=0 \implies f_0=a_0e^{-x}.$$

At order ε we have

$$f_1'+f_1+f_0''=0 \implies f_1=[a_1-a_0x]e^{-x}.$$

The second solution is singular, and the distinguished scaling (to balance the first two terms) is $\delta=\varepsilon$. We introduce a new variable $z=(x-a)/\varepsilon$ to have

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}+\frac{\mathrm{d}f}{\mathrm{d}z}+\varepsilon f=0$$

with solution

$$f=F_0(z)+\varepsilon F_1(z)+\cdots$$

At order 1 we have

$$F_0''+F_0'=0 \implies F_0'=-B_0e^{-z}$$

$$F_0(z)=A_0+B_0e^{-z}.$$

At order ε we have

$$F_1''+F_1'+F_0=0 \implies F_1'=-A_0-B_0ze^{-z}-B_1e^{-z}$$

$$F_1=A_1-A_0z+B_0[ze^{-z}+e^{-z}]+B_1e^{-z}.$$

We now have two possible solutions:

$$\begin{aligned} f(x) &\sim a_0e^{-x}+\varepsilon[a_1-a_0x]e^{-x}+\cdots \\ F(z) &\sim A_0+B_0e^{-z}+\varepsilon[A_1-A_0z+B_0(ze^{-z}+e^{-z})+B_1e^{-z}]+\cdots \end{aligned}$$

Question: Will we ever need to use both of these in the same problem?

Answer: The short answer is yes. This is a second-order differential equation, so we are entitled to demand that the solution satisfies two boundary conditions.

Suppose, with the differential equation above, the boundary conditions are

$$f=e^{-1} \text{ at } x=1 \qquad \text{and} \qquad \frac{\mathrm{d}f}{\mathrm{d}x}=0 \text{ at } x=0.$$

We will start by assuming that the unstretched form will do, and apply the boundary condition at $x=1$ to it:

$$f(x)\sim a_0e^{-x}+\varepsilon[a_1-a_0x]e^{-x}+\cdots$$

$$e^{-1} = a_0 e^{-1} + \varepsilon[a_1 - a_0]e^{-1} + \cdots$$

which immediately yields the conditions $a_0 = 1$, $a_1 = 1$. If we had continued to higher orders we would be able to find the constants there as well.

Now what about the other boundary condition? We have no more disposable constants so we'd be very lucky if it worked! In fact we have

$$f'(x) = -a_0 e^{-x} + \varepsilon[-a_1 - a_0 + a_0 x]e^{-x} + \cdots$$

so at $x = 0$,

$$f'(0) = -1 - 2\varepsilon + \cdots$$

This is where we have to use the other solution. If we fix $a = 0$ in the scaling for z, then the strained region is near $x = 0$. We can re-express the boundary condition in terms of z:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = 0 \text{ at } z = 0.$$

Now applying this boundary condition to our strained expansion gives:

$$F(z) \sim A_0 + B_0 e^{-z} + \varepsilon[A_1 - A_0 z + B_0(ze^{-z} + e^{-z}) + B_1 e^{-z}] + \cdots$$

$$F'(z) = -B_0 e^{-z} + \varepsilon[-A_0 - B_0 z e^{-z} - B_1 e^{-z}] + \cdots$$

and at $z = 0$,

$$F'(0) = -B_0 + \varepsilon[-A_0 - B_1] + \cdots$$

Imposing $F'(0) = 0$ fixes $B_0 = 0$, $B_1 = -A_0$ but does not determine A_0, B_1 or A_1. The solution which matches the $x = 0$ boundary condition is

$$F(z) \sim A_0 + \varepsilon[A_1 - A_0 z - A_0 e^{-z}] + \cdots$$

We now have two perturbation expansions, one valid at $x = 1$ and for most of our region, the other valid near $x = 0$. We have not determined all our parameters. How will we do this? The answer is **matching**.

6.1 Intermediate variable

Suppose (as in the example above) we have two asymptotic solutions to a given problem.

- One scales normally and satisfies a boundary condition somewhere away from the tricky region: we will call this the **outer** solution.
- The other is expressed in terms of a scaled variable, and is valid in a narrow region, (probably) near the other boundary. We will call this the **inner** solution.

In order to make sure that these two expressions both belong to the same real (physical) solution to the problem, we need to match them.

In the case where the outer solution is

$$f(x) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \cdots$$

and the inner

$$F(z) = F_0(z) + \varepsilon F_1(z) + \varepsilon^2 F_2(z) + \cdots$$

with scalings $z = x/\varepsilon$, we will match the two expressions using an **intermediate variable**. This is a new variable, ξ, intermediate in size between x and z, so that when ξ is order 1, x is small and z is large. We can define it as

$$x = \varepsilon^\alpha \xi \implies z = \varepsilon^{\alpha-1}\xi,$$

for α between 0 and 1. It is best to keep α symbolic[5].

The procedure is to substitute ξ into both $f(x)$ and $F(z)$ and then collect orders of ε and force the two expressions to be equal. This is best seen by revisiting the previous example.

Example continued

We had

$$f(x) = e^{-x} + \varepsilon(1-x)e^{-x} + \cdots$$

and

$$F(z) = A_0 + \varepsilon[A_1 - A_0 z - A_0 e^{-z}] + \cdots$$

with $z = x/\varepsilon$. Defining $x = \varepsilon^\alpha \xi$, we look first at $f(x)$:

$$f(x) = e^{-\varepsilon^\alpha \xi} + \varepsilon(1-\varepsilon^\alpha \xi)e^{-\varepsilon^\alpha \xi} + \cdots$$

Since $\varepsilon^\alpha \ll 1$ we can expand the exponential terms to give

$$f(x) = 1 - \varepsilon^\alpha \xi - \frac{1}{2}\varepsilon^{2\alpha}\xi^2 + \varepsilon - 2\varepsilon^{\alpha+1}\xi + O(\varepsilon^2, \varepsilon^{1+2\alpha}, \varepsilon^{3\alpha})$$

Now we look at $F(z)$. Note that $z = \varepsilon^{\alpha-1}\xi$, which is large.

$$F(z) = A_0 + \varepsilon[A_1 - A_0\varepsilon^{\alpha-1}\xi - A_0 e^{-\varepsilon^{\alpha-1}\xi}] + \cdots$$

Here the exponential terms become very small indeed so we neglect them and have

$$F(z) = A_0 - A_0\varepsilon^\alpha \xi + \varepsilon A_1 + \cdots$$

Comparing terms of the two expansions, at order 1 we have

$$1 = A_0$$

and at order ε^α,

$$-\xi = -A_0\xi$$

which is automatically satisfied if $A_0 = 1$. If we fix $\alpha > 1/2$ then the next term is order ε, giving

$$1 = A_1.$$

The next term in the outer expansion is order $\varepsilon^{2\alpha}$, but to match that we would have to go to order ε^2 in the inner expansion.

[5]However, occasionally you may find it quicker to pick a value of $\alpha = 1/2$, say. Be warned: sometimes there is only a specific range of α which works.

We have now determined all the constants to this order: so in the outer we have

$$f(x) = e^{-x} + \varepsilon(1 - x)e^{-x} + \cdots$$

and in the inner $x = \varepsilon z$,

$$F(z) = 1 + \varepsilon[1 - z - e^{-z}] + \cdots$$

Note: The beauty of the intermediate variable method for matching is that it has so much structure. If you have made any mistakes in solving either inner or outer equation, or if (by chance) you have put the inner region next to the wrong boundary, the structure of the two solutions won't match and you will know something is wrong!

6.2 Where is the boundary layer?

In the last example we **assumed** the boundary layer would be next to the lower boundary.

If we didn't know, how would we work it out?

Let's start by trying the previous example, but attempting to put the boundary layer near $x = 1$.

Recall we had an outer solution:

$$f(x) \sim a_0 e^{-x} + \varepsilon[a_1 - a_0 x]e^{-x} + \cdots$$

and an inner solution

$$F(z) \sim A_0 + B_0 e^{-z} + \varepsilon[A_1 - A_0 z + B_0(ze^{-z} + e^{-z}) + B_1 e^{-z}] + \cdots$$

with $z = (x - a)/\varepsilon$.

This time we will try to fit the outer solution to the boundary condition at $x = 0$. We have

$$\frac{\mathrm{d}f}{\mathrm{d}x} \sim -a_0 e^{-x} + \varepsilon[a_0 x - a_0 - a_1]e^{-x} + \cdots$$

so the condition is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 0 \qquad \text{at} \qquad x = 0.$$
$$0 = -a_0 + \varepsilon[-a_1 - a_0] + \cdots$$

which gives $a_0 = 0$, $a_1 = 0$ and so on. It is clear that we're not going to get a solution this way!

However, there is another problem, which appears when we try to fit the inner solution at the other boundary. We are setting $a = 1$ and trying to fit $F(z) = e^{-1}$ at $z = 0$. This gives:

$$e^{-1} = A_0 + B_0 + \varepsilon[A_1 + B_0 + B_1] + \cdots$$

so $A_0 = e^{-1} - B_0$ and $A_1 = -B_0 - B_1$. This seems fine, but look at the solution we get:

$$F(z) \sim e^{-1} + B_0(e^{-z} - 1) + \varepsilon[-e^{-1}z + B_0(z - 1 + (z+1)e^{-z}) + B_1(e^{-z} - 1)] + \cdots$$

Remember that, now the boundary layer is at the top, the outer limit of the inner solution will be for large **negative** z: in other words, all of these exponentials will be growing! This can never match onto a well-behaved outer solution.

Key fact: The boundary layer is always positioned so that any exponentials in the inner solution **decay** as you move towards the outer.

6.3 Linear example

This comes from Hinch (and originally, Friedricks). Consider:

$$\varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} = 1 + 2x \qquad \text{in } \ 0 < x < 1$$

with boundary conditions $f(0) = 0$ and $f(1) = 1$.

First we look for stretches that work (note that because the equation is linear, there is no mileage in scaling f). The right hand side of the equation is always strictly order 1 in our range of x, so if we stretch x as $x = a + \delta X$ we have three terms to compare:

$$[\mathbf{A}] \ \ \varepsilon\delta^{-2} \qquad [\mathbf{B}] \ \ \delta^{-1} \qquad [\mathbf{C}] \ \ 1.$$

For very small δ we have $[\mathbf{A}] \gg [\mathbf{B}] \gg [\mathbf{C}]$, and $[\mathbf{B}]$ catches $[\mathbf{A}]$ when $\delta = \varepsilon$. Then $[\mathbf{C}]$ catches $[\mathbf{B}]$ at $\delta = 1$ (which is the largest value of δ we can use, given that the range of x is only order 1).

Thus there are two distnguished stretches: the original variable x and a stretched variable $x = a + \varepsilon z$. Let us look at the regular, outer, solution first. Since we don't yet know where to put the boundary layer we won't use any boundary conditions on it yet.

We pose an expansion

$$f \sim f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$

and have

$$\begin{array}{ccccc} & & f_0' & = & 1 + 2x \\ \varepsilon f_0'' & + & \varepsilon f_1' & = & 0 \\ \varepsilon^2 f_1'' & + & \varepsilon^2 f_2' & = & 0 \end{array}$$

Integrating these in turn gives:

Order 1 $f_0' = 1 + 2x$ so $f_0 = x + x^2 + a_0$.

Order ε $f_1' = -2$ so $f_1 = -2x + a_1$.

Order ε^2 $f_2' = 0$ so $f_2 = a_2$.

Our regular expansion is

$$f \sim a_0 + x + x^2 + \varepsilon(a_1 - 2x) + \varepsilon^2 a_2 + \cdots$$

Now we move onto the inner, stretched solution. Recasting the ODE in terms of z gives

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + \frac{\mathrm{d}f}{\mathrm{d}z} = \varepsilon(1 + 2a) + 2\varepsilon^2 z.$$

We pose our expansion:

$$f \sim F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots$$

to have

$$\begin{array}{rcrcl} F_0'' & + & F_0' & = & 0 \\ \varepsilon F_1'' & + & \varepsilon F_1' & = & \varepsilon(1+2a) \\ \varepsilon^2 F_2'' & + & \varepsilon^2 F_2' & = & 2\varepsilon^2 z \end{array}$$

Solving the leading-order equation gives

$$F_0'' + F_0' = 0 \qquad F_0 = A_0 + B_0 e^{-z}.$$

Immediately our condition that any exponentials must decay outside the boundary layer tells us that the boundary layer is positioned near $x = 0$ (so that z is positive towards the interior of the domain) and thus $a = 0$. That means that we can apply to our inner solution the boundary condition $f(0) = 0$.

Order 1 We know $F_0 = A_0 + B_0 e^{-z}$, and applying the boundary condition gives $F_0 = A_0(1 - e^{-z})$.

Order ε $F_1'' + F_1' = 1$ gives $F_1 = A_1 + B_1 e^{-z} + z$, and the boundary condition forces $F_1 = A_1(1 - e^{-z}) + z$.

Order ε^2 $F_2'' + F_2' = 2z$ gives $F_2 = A_2 + B_2 e^{-z} + z^2 - 2z$, and the boundary condition forces $F_2 = A_2(1 - e^{-z}) + z^2 - 2z$.

Now we return to the outer solution, to which we can now apply the other boundary condition $f(1) = 1$:

$$1 \sim a_0 + 2 + \varepsilon(a_1 - 2) + \varepsilon^2 a_2 + \cdots$$

which fixes $a_0 = -1$, $a_1 = 2$ and $a_2 = 0$. We now have our two expansions:

$$f_{\text{outer}} \sim -1 + x + x^2 + \varepsilon(2 - 2x) + O(\varepsilon^3)$$

$$f_{\text{inner}} \sim A_0(1 - e^{-z}) + \varepsilon[A_1(1 - e^{-z}) + z] + \varepsilon^2[A_2(1 - e^{-z}) + z^2 - 2z] + \cdots$$

linked by the variables $x = \varepsilon z$.

To match the expansions, we introduce $\eta = \varepsilon^{-\alpha} x = \varepsilon^{1-\alpha} z$ and substitute in each:

$$\begin{array}{rcl} f_{\text{outer}} & = & -1 + \varepsilon^{\alpha}\eta + \varepsilon^{2\alpha}\eta^2 + 2\varepsilon - 2\varepsilon^{1+\alpha}\eta + O(\varepsilon^3) \\ f_{\text{inner}} & = & A_0 + \varepsilon^{\alpha}\eta + \varepsilon^{2\alpha}\eta^2 + \varepsilon A_1 - 2\varepsilon^{1+\alpha}\eta + \varepsilon^2 A_2 + \cdots \end{array}$$

Comparing terms at each order, we can immediately see that our expansions are succeeding in that some of the terms have already matched each other. To complete the match we need $A_0 = -1$, $A_1 = 2$ and $A_2 = 0$. Thus our two expansions are

$$f_{\text{outer}} \sim -1 + x + x^2 + \varepsilon(2 - 2x) + O(\varepsilon^3)$$

$$f_{\text{inner}} \sim e^{-z} - 1 + \varepsilon[2(1 - e^{-z}) + z] + \varepsilon^2[z^2 - 2z] + \cdots$$

linked via $x = \varepsilon z$.

Plotting these expansions for $\varepsilon = 0.1$, $\varepsilon = 0.03$ and $\varepsilon = 0.01$ shows the power of the method:

Here the outer expansions are the solid curves and the inner, the dashed curves. As ε gets smaller, the outer is a good approximation for a larger and larger proportion of the range, but the inner expansion is still needed near $x = 0$.

6.4 Another Example

This is a simplified version of an advection-diffusion problem that arose in my own research[6]. Solve

$$\frac{x}{y}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} + \frac{f}{y} - \varepsilon \nabla^2 f = 0$$

with boundary conditions

$$f + \varepsilon \frac{\partial f}{\partial y} = 0 \quad \text{at} \quad y = 1, \qquad\qquad f = 2 \quad \text{at} \quad y = 2$$

The boundary condition at $y = 1$ corresponds to a condition of **no flux** of f through the boundary $y = 1$.

Outer solution

We expand the PDE:

$$\frac{x}{y}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} + \frac{f}{y} - \varepsilon \frac{\partial^2 f}{\partial x^2} - \varepsilon \frac{\partial^2 f}{\partial y^2} = 0$$

and look for the first term of an outer solution by considering the case $\varepsilon = 0$:

$$\frac{x}{y}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} + \frac{f}{y} = 0 \qquad\quad x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} + f = 0$$

Because this is a first-order PDE we can apply the method of characteristics, solving:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x \qquad\quad \frac{\mathrm{d}y}{\mathrm{d}t} = -y$$

which gives us the parametric curves

$$x = x_0 e^t \qquad\quad y = e^{-t} \qquad\quad x = x_0/y$$

along which

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial f}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial f}{\partial y} = x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = -f$$

so

$$f = A(x_0)e^{-t} = A(xy)y.$$

We'll stay at one term for the outer solution.

[6] JFM, 534, 97–114, 2005

Scaling for the inner

We're expecting a boundary layer somewhere, because all the highest derivatives were neglected when we put $\varepsilon = 0$. In fact the type of the boundary condition at $y = 1$ gives us the hint: if $f + \varepsilon\partial f/\partial y = 0$ then $\partial f/\partial y$ must be an order of magnitude larger than f. So we look to scale y rather than x. (Another motivation for this choice is that, usually, boundary layers live near boundaries: and there are no boundaries on x.)

So we set $y = a + \varepsilon^b z$ and substitute in to the PDE:

$$\frac{x}{[a + O(\varepsilon^b)]}\frac{\partial f}{\partial x} - \varepsilon^{-b}\frac{\partial f}{\partial z} + \frac{f}{[a + O(\varepsilon^b)]} - \varepsilon\frac{\partial^2 f}{\partial x^2} - \varepsilon^{1-2b}\frac{\partial^2 f}{\partial z^2} = 0$$

Clearly the two terms which may be larger than $O(1)$ if $b > 0$ are the second and last terms: ε^{-b} and ε^{1-2b}. Balancing the two fixes $b = 1$ (which we expected from the boundary condition). Thus:

$$-\frac{\partial f}{\partial z} - \frac{\partial^2 f}{\partial z^2} + \varepsilon\frac{x}{a}\frac{\partial f}{\partial x} + \varepsilon\frac{f}{a} = O(\varepsilon^2).$$

Let's just look at the leading-order term first: $f = f_0 + \varepsilon f_1 + \cdots$ gives

$$-\frac{\partial f_0}{\partial z} - \frac{\partial^2 f_0}{\partial z^2} = 0 \qquad f_0 = A_0(x) + B_0(x)e^{-z}.$$

The exponential in z tells us that the boundary layer must be located at the lower boundary so $a = 1$ and $y = 1 + \varepsilon z$. Then we expect the outer to satisfy the upper boundary condition at $y = 2$; now we can return to the outer and complete it.

Full outer solution

We now have the outer solution

$$f = A(xy)y + \varepsilon f_1(x, y) + \cdots$$

which we need to satisfy the boundary condition $f(x, 2) = 2$. Applying this at leading order gives

$$2 = 2A(2x) \qquad A(\eta) = 1 \qquad f = y + \varepsilon f_1(x, y) + \cdots$$

Now we can continue with the expansion: the original equation was

$$x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} + f - y\varepsilon\frac{\partial^2 f}{\partial x^2} - y\varepsilon\frac{\partial^2 f}{\partial y^2} = 0$$

so we have

$$\begin{array}{ccccccccccc}
x\partial f_0/\partial x & - & y\partial f_0/\partial y & + & f & & & & & = & 0 \\
x\partial f_1/\partial x & - & y\partial f_1/\partial y & + & f_1 & - & y\partial^2 f_0/\partial x^2 & - & y\partial^2 f_0/\partial y^2 & = & 0 \\
x\partial f_2/\partial x & - & y\partial f_2/\partial y & + & f_2 & - & y\partial^2 f_1/\partial x^2 & - & y\partial^2 f_1/\partial y^2 & = & 0
\end{array}$$

with boundary conditions

$$f_0(x, 2) = 2 \qquad f_1(x, 2) = 0 \qquad f_2(x, 2) = 0$$

At order 1 we know this is satisfied by $f_0 = y$. At order ε we have

$$x\frac{\partial f_1}{\partial x} - y\frac{\partial f_1}{\partial y} + f_1 = 0$$

which is the same equation we had for f_0, so has solution $f_1 = A_1(xy)y$. This time the boundary condition gives $f_1 = 0$. We can see that this pattern will continue, and in fact $f_n = 0$ for $n \geq 1$: the full outer solution is

$$f_{\text{outer}} = y.$$

Full inner solution

We now return to the inner solution:

$$\frac{\partial^2 f}{\partial z^2} + \frac{\partial f}{\partial z} = \varepsilon\frac{x}{[1+\varepsilon z]}\frac{\partial f}{\partial x} + \varepsilon\frac{f}{[1+\varepsilon z]} - \varepsilon^2\frac{\partial^2 f}{\partial x^2}$$

Keeping terms up to order ε gives

$$\frac{\partial^2 f_0}{\partial z^2} + \frac{\partial f_0}{\partial z} = 0$$

$$\frac{\partial^2 f_1}{\partial z^2} + \frac{\partial f_1}{\partial z} = x\frac{\partial f_0}{\partial x} + f_0$$

with boundary conditions (true at each order)

$$f + \partial f/\partial z = 0 \qquad \text{at } z = 0$$

At order 1 we have

$$f_0 = A_0(x) + B_0(x)e^{-z}$$

and the boundary condition gives $A_0(x) = 0$: $f_0 = B_0(x)e^{-z}$.
At order ε we have

$$\frac{\partial^2 f_1}{\partial z^2} + \frac{\partial f_1}{\partial z} = [xB_0' + B_0]e^{-z}$$

which gives

$$f_1 = A_1(x) + B_1(x)e^{-z} - [xB_0' + B_0]ze^{-z}.$$

Applying the boundary condition fixes $A_1(x) = [xB_0' + B_0]$. Thus our solution (to order ε) is

$$f = B_0(x)e^{-z} + \varepsilon[(xB_0'(x) + B_0(x))(1 - ze^{-z}) + B_1(x)e^{-z}] + O(\varepsilon^2).$$

Matching

Our two solutions are:

$$f_{\text{outer}} = y.$$

$$f_{\text{inner}} = B_0(x)e^{-z} + \varepsilon[(xB_0'(x) + B_0(x))(1 - ze^{-z}) + B_1(x)e^{-z}] + O(\varepsilon^2).$$

Using an intermediate variable $y = 1 + \varepsilon^\alpha \eta$, $z = \varepsilon^{\alpha-1}\eta$, the outer becomes

$$f_{\text{outer}} = 1 + \varepsilon^\alpha \eta$$

and the inner (neglecting decaying exponentials)

$$f_{\text{inner}} = \varepsilon(xB_0'(x) + B_0(x)) + O(\varepsilon^2).$$

There is nothing in the inner large enough to match onto the 1 in the outer.

However, remember we started from a **linear** equation. Along with the fact that the inner boundary condition was homogeneous, that means that if f_{inner} is a solution, so is $\varepsilon^{-1} f_{\text{inner}}$. So we try that:

$$\begin{aligned} f_{\text{inner,new}} &= \varepsilon^{-1}B_0(x)e^{-z} + (xB_0'(x) + B_0(x))(1 - ze^{-z}) + B_1(x)e^{-z} + O(\varepsilon) \\ &\sim (xB_0'(x) + B_0(x)) + O(\varepsilon) \quad \text{as } z \to \infty. \end{aligned}$$

Now we can match the two functions if

$$xB_0'(x) + B_0(x) = 1$$

which is just an ODE. Solving gives $B_0(x) = 1 + C/x$ and since the line $x = 0$ is within our domain, we require $C = 0$ for regularity. Thus:

$$\begin{aligned} f_{\text{outer}} &= y \\ f_{\text{inner}} &= \varepsilon^{-1}e^{-z} + (1 - ze^{-z}) + B_1(x)e^{-z} + O(\varepsilon) \end{aligned}$$

with $y = 1 + \varepsilon z$. To determine B_1 we would have to calculate the f_2 term of the inner expansion.

F Laplace's equation: Complex variables

Let's look at Laplace's equation in 2D, using Cartesian coordinates:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

It has no real characteristics because its discriminant is negative ($B^2 - 4AC = -4$). But if we ignore this technicality and allow ourselves a complex change of variables, we can benefit from the same structure of solution that worked for the wave equation. Introduce

$$\begin{aligned} \eta &= x + iy & \qquad x &= (\eta + \xi)/2 \\ \xi &= x - iy & \qquad y &= (\eta - \xi)/2i. \end{aligned}$$

Then the chain rule gives

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)$$

and the PDE becomes

$$4\frac{\partial}{\partial \eta}\frac{\partial f}{\partial \xi} = 0$$

whose solution is straightforward:

$$f = p(\eta) + q(\xi) = p(x + iy) + q(x - iy).$$

Here p and q are differentiable complex functions; and assuming we wanted a real solution to the original (real) PDE, we have an additional constraint that the sum of the two functions must have no imaginary part.

We can formalise this in more standard notation: if we use the (x, y) plane to represent the complex plane in the usual way, we introduce the complex variable $z = x + iy$. Then its complex conjugate is $\overline{z} = x - iy$ and the solution we have just found is

$$f = p(z) + q(\overline{z}).$$

F.1 Cauchy-Riemann Equations

Let's look at our function $p(\eta) = p(z)$, which forms half of our "characteristics"-style solution. It is obvious that

$$\frac{\partial p}{\partial \xi} = \frac{\partial p}{\partial \overline{z}} = 0$$

and using the chain rule, this tells us that

$$\frac{1}{2}\frac{\partial p}{\partial x} - \frac{1}{2i}\frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial x} = -i\frac{\partial p}{\partial y}.$$

Now if we divide the function into its real and imaginary parts:

$$p(z) = u(x, y) + iv(x, y)$$

where u and v are real functions, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

This complex equation is equivalent to the pair of real equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the **Cauchy-Riemann equations**, and are satisfied by the real and imaginary parts of any differentiable function of a complex variable $z = x + iy$.

In fact in a given domain, u and v (continuously differentiable) satisfy the Cauchy-Riemann equations if and only if p is an **analytic** function of z. We will not prove this here.

(Recall $f(z)$ is analytic $\equiv$ holomorphic within a domain D if, in every circle $|z - z_1| < \rho$ lying in D, f can be represented as a power series in $z - z_1$.)

F.2 General solution of Laplace's equation

We had the solution

$$f = p(z) + q(\overline{z})$$

in which $p(z)$ is analytic; but we can go further: remember that Laplace's equation in 2D can be written in polar coordinates as

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$$

and we showed by separating variables that in the whole plane (except the origin) it has solutions

$$f(r,\theta) = A + B\ln r + \int r^{\lambda}(a(\lambda)\cos[\lambda\theta] + b(\lambda)\sin[\lambda\theta])\,\mathrm{d}\lambda$$
$$+ \int e^{\mu\theta}(c(\mu)\cos[\mu\ln r] + d(\mu)\sin[\mu\ln r])\,\mathrm{d}\mu. \quad (7)$$

Now in these variables, $z = r\exp[i\theta]$ so we can also write the solution we found as

$$f = \mathrm{Real}\left(A + B\ln z + \int \overline{a}(\lambda)z^{\lambda}\,\mathrm{d}\lambda + \int \overline{c}(\mu)z^{i\mu}\,\mathrm{d}\mu\right)$$

meaning that our solution is the real part of a function of z only:

$$f = \mathrm{Real}\,(g(z)).$$

Note that $g(z)$ as given here is analytic in any domain that does not encircle the origin; to make it analytic in a general domain we need the additional constraints that $\overline{c}(\lambda) = 0$, $B = 0$ and $\overline{a}(\lambda)$ is only nonzero for integer values of λ.

We have shown that the real solution to Laplace's equation we had found is the real part of an analytic function of $z = x + iy$ in our domain; we can show the converse very quickly from the Cauchy-Riemann equations. Consider an analytic function

$$f(z) = u(x,y) + iv(x,y)$$

Then the Cauchy-Riemann equations give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Differentiating the first w.r.t. x and the second w.r.t. y gives:

$$\frac{\partial^2 v}{\partial x\partial y} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We can solve Laplace's equation in any domain simply by taking the real part of **any analytic function** in that domain.

F.3 Composition of Analytic functions

The composition of two analytic functions is analytic (providing, of course, the relevant domains are correctly specified): if

$$f : D_1 \to D_2 \qquad \text{and} \qquad g : D_2 \to D_3$$

are both analytic, then the composed function

$$g \circ f : D_1 \to D_3$$

is also analytic on D_1.

This has important ramifications for the solution of Laplace's equation in odd-shaped domains or with boundary conditions which are unsuitable for separation of variables.

Suppose we are trying to find a real function u satisfying

$$\nabla^2 u = 0 \ \text{ in } \ D_1 \qquad\quad \text{with } \ u = u(x,y) \ \text{ on } \ \partial D_1.$$

Of course this is equivalent to finding an analytic function $f(z)$ on D_1 whose real part satisfies the boundary condition on ∂D_1.

If D_1 is an awkward shape, and we can find an analytic function $w(z)$ which maps it to a more helpful domain D_2, then we can define

$$f = g \circ w \qquad\quad f(z) = g(w(z))$$

and we are now looking for an analytic function g defined on D_2 such that

$$\text{Real}\,(g(w(z))) = u(z) \ \text{ on } \ \partial D_1.$$

$$\text{Real}\,(g(w)) = u(z(w)) \ \text{ on } \ \partial D_2.$$

Example

This is taken from an old UCL exam paper.

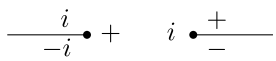
Find the solution to Laplace's equation in the domain D_1 given by the whole (x, y)-plane except for two semi-infinite plates $|x| \geq 1$, $y = 0$. The boundary conditions on these two plates are

$$u(x,0) = 0 \quad \text{on} \ \ x \geq 1; \qquad\quad u(x,0) = 1 \quad \text{on} \ \ x \leq -1.$$

The domain looks superficially suitable for separation of variables in Cartesian coordinates, but the boundary conditions are not suitable: we would need $u(x, 0)$ to be prescribed for **all** x for separation to work.

Here we use the map $w(z) = z + \sqrt{(z^2 - 1)}$. Note that the square root means this map is not analytic over the whole plane; we need a branch cut at each of $z = 1$, $z = -1$. Given the domain we are trying to transform, it makes sense to put the branch cuts on $y = 0$ (or z real) and $|x| \geq 1$ (or $|z| \geq 1$).

The point $z = 0$ maps to $w = \sqrt{(-1)}$ and we can choose which of the possible values we take for the sign of the square root here: we choose $w(0) = i$. This choice, with the positioning of the branch cuts, determines $w(z)$ everywhere in our domain – in the diagram I've marked the result of each of the square roots at points around it. So when $z = x + i\varepsilon$ and $x > 1$, both roots are positive; when $z = x$ and $|x| < 1$, the root at $z = 1$ has argument i and the other is positive; when $z = x - i\varepsilon$ with $x < -1$, both roots have argument i so the product is negative, and so on:



In particular:

$$\begin{array}{ll}
w(-1) = -1 & w(1) = 1 \\
w(x) = x + i\sqrt{(1 - x^2)} & -1 < x < 1 \\
w(x + i\varepsilon) = x - \sqrt{(x^2 - 1)} & x < -1 \\
w(x - i\varepsilon) = x + \sqrt{(x^2 - 1)} & x < -1 \\
w(x + i\varepsilon) = x + \sqrt{(x^2 - 1)} & x > 1 \\
w(x - i\varepsilon) = x - \sqrt{(x^2 - 1)} & x > 1
\end{array}$$

Thus the branch cuts in the z-plane map onto the real line in the w-plane: the left-hand cut maps to (top side) $w < -1$ and (bottom side) $-1 < w < 0$, and the right-hand cut maps to (top side) $w > 1$ and (bottom side) $0 < w < 1$. In the w-plane we now need to solve Laplace's equation for a new function v with

$$v(x,0) = 1 \quad \text{on} \quad x \le 0 \qquad\qquad v(x,0) = 0 \quad \text{on} \quad x \ge 0.$$

This new problem is suitable for separation of variables in polar coordinates: the boundary conditions in terms of r and θ are

$$v(r,0) = 0 \qquad\qquad v(r,\pi) = 1.$$

Note that our domain now does not encircle the origin, so we must revisit our separable solution and include some terms we discarded earlier.

We look for the form $v = R(r)T(\theta)$ and derive the coupled ODEs

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = A \qquad\qquad \frac{T''(\theta)}{T(\theta)} = -A.$$

In the three cases $A < 0$, $A > 0$ and $A = 0$ respectively these yield:

$$v = (A_\mu \exp[\mu\theta] + B_\mu \exp[-\mu\theta])(C_\mu \cos[\mu \ln r] + D_\mu \sin[\mu \ln r])$$
$$v = (a_\lambda \cos[\lambda\theta] + b_\lambda \sin[\lambda\theta])(c_\lambda r^\lambda + d_\lambda r^{-\lambda})$$
$$v = (\alpha + \beta \ln r)(\gamma + \delta\theta).$$

Applying the boundary condition $T(\theta = 0) = 0$ gives the three basis functions

$$v = \sinh[\mu\theta](C_\mu \cos[\mu \ln r] + D_\mu \sin[\mu \ln r])$$
$$v = \sin[\lambda\theta](c_\lambda r^\lambda + d_\lambda r^{-\lambda})$$
$$v = \theta(\alpha + \beta \ln r),$$

and the condition that v must be well-behaved at $r = 0$ (since the origin is in our domain) fixes further:

$$v = \alpha\theta + \sum_\lambda c_\lambda r^\lambda \sin[\lambda\theta]$$

The final boundary condition $v(r,\pi) = 1$ gives

$$1 = \alpha\pi + \sum_\lambda c_\lambda r^\lambda \sin[\lambda\pi]$$

which is satisfied with $\alpha = 1/\pi$ and $c_\lambda = 0$. Thus we have found

$$v(r,\theta) = \frac{\theta}{\pi}.$$

In order to convert this to a solution to our original problem, we first need to find the analytic function of which it is the real part. In this case the function is straightforward:

$$v(r,\theta) = \frac{\theta}{\pi} = -\frac{1}{\pi}\,\text{Real}\,(i[\ln r + i\theta]) = -\frac{1}{\pi}\,\text{Real}\,(i \ln w)$$

so the analytic function we need is

$$g(w) = -\frac{i \ln w}{\pi}.$$

Finally we need to convert back to the original variables:

$$f(z) = g \circ w(z) = -\frac{i}{\pi} \ln \left\{ z + \sqrt{(z^2 - 1)} \right\}$$

and the solution we need is the real part of this:

$$u(x, y) = \text{Real} \left(-\frac{i}{\pi} \ln \left\{ z + \sqrt{(z^2 - 1)} \right\} \right) = \frac{1}{\pi} \text{Imag} \left(\ln \left\{ z + \sqrt{(z^2 - 1)} \right\} \right).$$

In particular, on the "missing line" $y = 0$, $-1 \leq x \leq 1$, we have

$$u(x, 0) = \frac{1}{\pi} \text{Arg} \left(\left\{ x + i\sqrt{(1 - x^2)} \right\} \right) = \frac{1}{\pi} \arctan \frac{\sqrt{(1 - x^2)}}{x}.$$

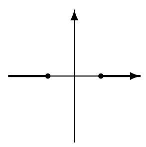
G Conformal maps

You will sometimes see these analytic functions referred to as conformal maps: in fact there is a subtle distinction. An analytic function provides a conformal map only if its derivative is nonzero throughout the domain. The meaning of *conformal map* is a map which preserves angles (though not necessarily lengths).

Most of the art of using conformal maps to improve problems involving Laplace's equation is in choosing the map to use. Here are a few domains and functions which improve them (starting with the example we just solved).

Two infinite plates

The geometry we just discussed took the plane with two branch cuts missing:



was transformed under the mapping

$$w(z) = z + \sqrt{(z^2 - 1)}$$

to the upper half plane.

Bilinear mapping

We can take the upper half plane $\text{Imag}\,(z) \geq 0$ to the unit disc $|w| \leq 1$ using

$$w(z) = \frac{z - \alpha}{z - \overline{\alpha}}$$

where α is any constant with $\text{Imag}\,(\alpha) > 0$.

Zhukovsky transform

In two-dimensional inviscid fluid mechanics with a steady velocity field (u, v), we can use a complex function

$$f(z) = \phi + i\psi \qquad z = x + iy$$

to represent the velocity:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = u + iv.$$

The Zhukovsky transform is one of the classics of early aerodynamics. Under the transformation:

$$w(z) = z + \frac{1}{z}$$

a disc not centred on the origin, whose boundary passes through the point $z = 1$, transforms into the Zhukovsky aerofoil:

Cardioid

The cardioid $r = 2(1 + \cos\theta)$:

can be transformed into the unit disc $|w| < 1$ using the mapping

$$w(z) = \sqrt{z} - 1$$

with the branch cut of the square root drawn along the negative x-axis.

G.1 Boundary conditions

Suppose we map domain D_1 of z into a more pleasant domain D_2 of the w-plane, using the mapping $w(z)$. Then it is clear that if we have boundary conditions of the form

$$u = f(z) \qquad \text{on } \partial D_1$$

they can be transferred to w simply by using the inverse mapping:

$$U = f(z(w)) \qquad \text{on } \partial D_2.$$

However, if our original boundary conditions were formed in terms of the derivative normal to the boundary:

$$\frac{\partial u}{\partial n} = g(z) \qquad \text{on } \partial D_1$$

we need to work a little harder. For the details of how to work this out, see Weinberger p. 243; but in essence, the derivatives normal to the boundary in the two domains are directly related via the (complex) derivative of the mapping $w(z)$:

$$\frac{\partial u}{\partial n} = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| \frac{\partial U}{\partial n} \qquad \text{so} \qquad \frac{\partial U}{\partial n} = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^{-1} g(z(w)) \qquad \text{on } \partial D_2.$$

7 The WKB method

The Wentzel, Kramers & Brillouin (WKB) method is a method for constructing approximate oscillatory solutions of differential equations that are varying over a fast and slow timescale. The standard WKB equation in one-dimensional (1-D) case is of the form

$$\frac{d^2\psi}{dx^2} + q(x)\psi = 0, \tag{1}$$

where $|q(x)|$ is "large". To emphasise that $|q(x)|$ is "large" (and let's here assume it is positive), it is convenient to represent it as $q(x) = \frac{\mu^2(x)}{\varepsilon^2}$, where $\mu(x) \sim O(1)$ and $\varepsilon \ll 1$ is a small constant parameter. From this we obtain

$$\varepsilon^2 \frac{d^2\psi}{dx^2} + \mu^2(x)\psi = 0. \tag{2}$$

Now, if $\mu(x)$ equals a constant then the solutions would be $Ae^{\pm \mathrm{i}\mu x/\varepsilon}$, where A is some arbitrary constant representing the amplitude, which are same as (1-D) plane time-harmonic waves. The case of small ε thus corresponds to high frequencies $\omega = \mu c/\varepsilon$ or small wavelength $\lambda = 2\varepsilon\pi/\mu$. For this reason, the WKB method represents a so-called "high-frequency" approximation. The basic intuition behind the WKB method is that when $mu(x)$ varies "slowly", the solution still behaves "microscopically" as a rapidlly oscillating plane wave, with slowly (macroscopically) varying amplitude and other parameters.

We seek the WKB approximation in the form

$$\psi(x, \varepsilon) \approx Ae^{\mathrm{i}\tau(x)/\varepsilon}, \tag{3}$$

with A constant and $\varepsilon \ll 1$. We then obtain from (2)

$$\mathrm{i}\varepsilon^{-1}\tau''(x) - \varepsilon^{-2}(\tau'(x))^2 + \varepsilon^{-2}\mu^2(x) = 0. \tag{4}$$

Now seek $\tau(x) = \tau(x, \varepsilon)$ in the so-called *regular perturbation form*, i.e. as a series in increasing powers of small parameter ε:

$$\tau(x, \varepsilon) = \tau_0(x) + \varepsilon\tau_1(x) + \varepsilon^2\tau_2(x) + \ldots \quad \text{for } \omega \to \infty. \tag{5}$$

Substituting (5) into (4), to leading order, i.e. $O(\varepsilon^{-2})$, we have $-\varepsilon^{-2}(\tau_0'(x))^2 + \varepsilon^{-2}\mu^2(x) = 0$. Hence

$$\tau_0(x) = \pm \int_{x_0}^{x} \mu(x')dx'. \tag{6}$$

The next order terms, $O(\varepsilon^{-1})$ in (4), are then $\mathrm{i}\varepsilon^{-1}\tau_0''(x) - 2\varepsilon^{-1}\tau_0'(x)\tau_1'(x) = 0$. This integrates to

$$\tau_1(x) = \frac{\mathrm{i}}{2}\ln|\tau_0'(x)| + C_\pm = \frac{\mathrm{i}}{2}\ln|\mu(x)| + C_\pm,$$

where $C_\pm$ are arbitrary constants. So we have, to first order, that

$$\tau(x) = \pm \int_{x_0}^{x} \mu(x')dx' + \varepsilon\left\{\frac{\mathrm{i}}{2}\ln|\mu(x)| + C_\pm\right\}.$$

Hence, from (3), our solution (the WKB approximation) is

$$\psi(x,\varepsilon) \approx A\exp\left\{\pm\frac{\mathrm{i}}{\varepsilon}\int_{x_0}^{x}\mu(x')dx' - \frac{1}{2}\ln|\mu(x)| + \tilde{C}_\pm\right\} = \frac{\tilde{A}}{(\mu(x))^{1/2}}e^{\pm\frac{\mathrm{i}}{\varepsilon}\int_{x_0}^{x}\mu(x')dx'},$$

or

$$\psi(x,\varepsilon) \approx \frac{1}{(\mu(x))^{1/2}}\left[A\,e^{\frac{\mathrm{i}}{\varepsilon}\int_{x_0}^{x}\mu(x')dx'} + B\,e^{-\frac{\mathrm{i}}{\varepsilon}\int_{x_0}^{x}\mu(x')dx'}\right] =: \tilde{\psi}, \tag{7}$$

where A and B are arbitrary constants.

This soltion can be interpreted as a superposition of two waves, one travelling in positive x direction and one travelling in the negative x direction both rapidly oscillating solutions within the envelope $\pm A\mu^{-1/2}(x)$ moving to towards $x = +\infty$ and $x = -\infty$, respectively for increasing t. For μ constant (7) reproduces (two) plane time-harmonic waves.

To examine the range of validity of the WKB approximation, we substitute (7) back into the original equation (2). After some algebraic manipulation we obtain,

$$\frac{d^2\tilde{\psi}}{dx^2} + \frac{\mu^2(x)}{\varepsilon^2}\tilde{\psi} = \left[\frac{3}{4}\frac{(\mu'(x))^2}{(\mu(x))^2} - \frac{1}{2}\frac{\mu''(x)}{\mu(x)}\right]\tilde{\psi}.$$

The expression suggests that the approximation remains accurate so long as

$$\left[\frac{3}{4}\frac{(\mu'(x))^2}{(\mu(x))^2} - \frac{1}{2}\frac{\mu''(x)}{\mu(x)}\right] \ll \frac{\mu^2(x)}{\varepsilon^2},$$

or equivalently

$$\varepsilon^2\left[\frac{3}{4}\frac{(\mu'(x))^2}{(\mu(x))^4} - \frac{1}{2}\frac{\mu''(x)}{(\mu(x))^3}\right] \ll 1. \tag{8}$$

Thus the wave number $\frac{\mu(x)}{\varepsilon}$ must be varying slowly enough in x and/or the frequency large enough for (8) to hold. Also, for (8) to hold μ should not be too close to zero: WKB solution indeed becomes infinite and the approximation fails whenever $\mu(x) = 0$. These critical values are called *turning points*. The name indicates that a wave fails to propagate beyond these points, "turning around", and alternative approximations are required in the neighbourhood of such points.

8 Integrals and Steepest Descents

8.1 Small parameter in the integration limits

Suppose you need to understand the asymptotic behaviour of

$$I(\rho) = \int_\rho^\infty \frac{e^{-\tau}}{\tau^2}\,\mathrm{d}\tau$$

for small ρ. How do we determine it? The procedure is actually quite straightforward: two steps of integration by parts give

$$I(\rho) = \frac{e^{-\rho}}{\rho} - \int_\rho^\infty \frac{e^{-\tau}}{\tau}\,\mathrm{d}\tau = \frac{e^{-\rho}}{\rho} + e^{-\rho}\ln\rho - \int_\rho^\infty e^{-\tau}\ln\tau\,\mathrm{d}\tau$$

and the last integral would now be convergent even if $\rho = 0$, when the answer is minus Euler's constant, γ. So we can move the integration:

$$I(\rho) = \frac{e^{-\rho}}{\rho} + e^{-\rho}\ln\rho + \gamma + \int_0^\rho e^{-\tau}\ln\tau\,\mathrm{d}\tau$$

Now the last integral is better behaved: τ is small throughout and the integral is uniformly convergent. That gives us the freedom to expand the exponential as a power series, and to interchange the order of integration and summation:

$$\int_0^\rho e^{-\tau}\ln\tau\,\mathrm{d}\tau = \int_0^\rho \ln\tau \sum_{n=0}^{\infty}\frac{(-\tau)^n}{n!}\,\mathrm{d}\tau = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\rho \tau^n\ln\tau\,\mathrm{d}\tau$$

We can integrate by parts again, but this time differentiating the log:

$$\int_0^\rho e^{-\tau}\ln\tau\,\mathrm{d}\tau = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\left[\ln\tau\frac{\tau^{n+1}}{(n+1)}\right]_0^\rho - \int_0^\rho\frac{\tau^n}{(n+1)}\,\mathrm{d}\tau\right)$$

The final integration step gives

$$\begin{aligned}\int_0^\rho e^{-\tau}\ln\tau\,\mathrm{d}\tau &= -\sum_{n=0}^{\infty}\frac{(-\rho)^{n+1}}{(n+1)!}\left(\ln\rho - \frac{1}{(n+1)}\right)\\ &= \ln\rho(1-e^{-\rho}) + \sum_{n=0}^{\infty}\frac{(-\rho)^{n+1}}{(n+1)!(n+1)}\end{aligned}$$

and so

$$I(\rho) = \frac{e^{-\rho}}{\rho} + \ln\rho + \gamma + \sum_{n=0}^{\infty}\frac{(-\rho)^{n+1}}{(n+1)!(n+1)}$$

Thus expanding the exponential and the sum as power series in ρ we obtain the asymptotic series we used earlier:

$$I(\rho) = \frac{1}{\rho} + \ln\rho + \gamma - 1 - \frac{\rho}{2} + \frac{\rho^2}{12} - \frac{\rho^3}{72} + O(\rho^4).$$

8.2 Small parameter in the integrand

We'll look now at a specific class of integral, which may seem rather limited: but these problems crop up in many applications, and particularly in finding the outer limit of an inner boundary-layer solution. We'll use an example.

We want to know the asymptotic behaviour as $z \to \infty$ of the real part of the integral

$$I(z) = \int_{-\infty}^{\infty}\exp\left[zf(t)\right]g(t)\ \mathrm{d}t$$

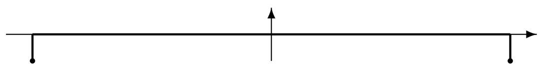
where $f(t)$ and $g(t)$ are analytic functions of t. Our example integral J has $f(t) = -t^2 + 2it + 1$ and $g(t) = 1$, so that

$$\mathrm{Real}\,(J(z)) = \int_{-\infty}^{\infty}\exp\left[-z(t^2-1)\right]\cos\left[2zt\right]\ \mathrm{d}t$$

The reason I've chosen these particular functions is that we can solve:

$$J(z) = \int_{-\infty}^{\infty} \exp\left[-z(t-i)^2\right] \, \mathrm{d}t = \int_{-\infty-i}^{\infty-i} \exp\left[-zu^2\right] \mathrm{d}u$$

and choosing the right integration contour in the u plane:



the contribution from the contour ends is exponentially small and we have

$$J(z) = \int_{-\infty}^{\infty} \exp\left[-zu^2\right] \mathrm{d}u = \left(\frac{\pi}{z}\right)^{1/2}.$$

How do we tackle the integral $I(z)$ in general? The obvious estimate would be to take t_0, the place on the integration contour at which Real $(f(t))$ is maximum and expand about it:

$$\begin{aligned}
I(z) &\sim \int_{-\infty}^{\infty} \exp\left[z(f(t_0) + (t-t_0)^2 f''(t_0)/2 + \cdots\right] g(t) \, \mathrm{d}t \\
&\sim \exp\left[zf(t_0)\right] g(t_0) \int_{-\infty}^{\infty} \exp\left[z(t-t_0)^2 f''(t_0)/2 + \cdots\right] \, \mathrm{d}t \\
&\sim \exp\left[zf(t_0)\right] g(t_0) (-zf''(t_0))^{-1/2}
\end{aligned}$$

However, this is a massive overestimate. Look at our example. The real part of $f(t)$ is $-t^2+1$, which is maximal at $t=0$, giving the estimate

$$J(z) \sim (2z)^{-1/2} e^z$$

which is too large by an exponential factor. The reason for this massive overestimate is that we have not allowed for the effect of $\cos[2zt]$, which oscillates very rapidly when z is large and causes a large cancellation effect. This cosine term came from the imaginary part of $f(t)$.

The technique which will save us is fundamentally based on the analyticity of $f(t)$ and $g(t)$. Because both they and the exponential function are analytic, we can deform our integration contour in the complex plane without changing the result. We deform our contour to get the **lowest** possible estimate for our function: and because in doing so we will avoid these cancellations which caused our other estimates to be too large, we will attain the best estimate for the integral.

We begin by observing that because $f(z)$ is analytic, its real part satisfies Laplace's equation, and thus the real part can have no maxima or minima, but only saddle points. Suppose we contour the real part of $f(t)$ over the whole t-plane: (these contours are for $J(z)$)

Then we can deform the contour so that it keeps Real $(f(t))$ as small as possible throughout: this will involve the contour passing over a saddle point, where the maximum value of Real $(f(t))$ will be attained.

Now we can choose as our contour the route over the saddle point which keeps $\text{Imag}\,(f(t))$ constant along it, and "correct" back to the ends of our original integration contour in the far field, along lines of constant $\text{Real}\,(f(t))$. The contribution from the correction curves will be asymptotically negligible: not only will $\text{Real}\,(f(t))$ be much smaller there than near the saddle, but also $\text{Imag}\,(f(t))$ is varying rapidly, giving the cancellation which was so troublesome earlier.

On the remaining integration contour, the imaginary part of $f(t)$ is constant so there will be no cancellation from rapid oscillations. Now it is appropriate to Taylor-expand $f(t)$ about the saddle point, and we can take as many terms as necessary to obtain our asymptotic expansion.

The contour which keeps the imaginary part of f constant is the same contour which makes the real part change most rapidly: thus we have chosen the contour of **steepest descent**.

For our example, $f(t) = (t-i)^2$ which has a saddle point at $t = i$. If we set $t = x+iy$, the imaginary part of f is $2x(y-1)$ which is zero at the saddle point, so we choose a contour which keeps it zero: $y = 1$. In fact this is exactly the procedure we carried out to calculate the integral – but as a method this works even when it doesn't exactly solve the problem.

Example: Bessel function

The Bessel function $K_\nu(z)$ can be written

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} \exp\left[\nu t - z\cosh t\right]\,\mathrm{d}t.$$

We will take $z = 1$, double the function and look for an expansion as $\nu \to \infty$:

$$I(\nu) = \int_{-\infty}^{\infty} \exp\left[\nu t - \cosh t\right]\,\mathrm{d}t.$$

If we generalise slightly to allow $f(t)$ to have ν as a parameter, we can write

$$I(\nu) = \int_{-\infty}^{\infty} \exp\left[\nu(t - \cosh t/\nu)\right]\,\mathrm{d}t.$$

Since there is no trigonometric term, we know that the imaginary part of our function is zero along the real line: so in this case we will not need to find a new contour, just find the dominant contribution from the contour we started from.

We look at the behaviour of $f(t) = t - \cosh t/\nu$. It has a maximum where

$$\frac{\mathrm{d}}{\mathrm{d}t}(t - \cosh t/\nu) = 0 \qquad\qquad \sinh t = \nu$$

and about that point, we expand:

$$\begin{aligned} f(t) &\approx f(\operatorname{arcsinh}\nu) + \frac{1}{2}(t - \operatorname{arcsinh}\nu)^2 f''(\operatorname{arcsinh}\nu) + \cdots \\ &= \operatorname{arcsinh}\nu - \frac{(\nu^2+1)^{1/2}}{\nu} - (t - \operatorname{arcsinh}\nu)^2\frac{(\nu^2+1)^{1/2}}{2\nu} + \cdots \end{aligned}$$

The contributions to the integral become

$$I(\nu) \sim \int_{-\infty}^{\infty} \exp\left[\nu\left(\operatorname{arcsinh}\nu - \frac{(\nu^2+1)^{1/2}}{\nu} - (t - \operatorname{arcsinh}\nu)^2 \frac{(\nu^2+1)^{1/2}}{2\nu}\right)\right] \mathrm{d}t$$

$$I(\nu) \sim \frac{(2\pi)^{1/2}}{(\nu^2+1)^{1/4}} \exp\left[\nu \operatorname{arcsinh}\nu - (\nu^2+1)^{1/2}\right].$$

Steepest descents in practice

Although we've used a contour which keeps the imaginary part of f constant, and thereby makes the real part decrease as fast as possible away from the contour, this level of detail is not necessary for the leading-order term of an asymptotic expansion. In practice, any contour which crosses the saddle and descends on both sides of it will do a decent job. Numerically, it is often less expensive to carry out the integration with smaller step sizes to catch the oscillation terms than it is to find the steepest descent contour exactly. So the key point is to find the saddle (or the highest saddle) over which your contour must pass: beyond that point you have much more flexibility.

9 More matching!

In section 6 we looked at matched asymptotic expansions in the situation where we found all the possible underlying scalings first, located where to put the boundary later from the direction of the exponential decay, applied all sets of boundary conditions and finally matched our two expansions. That's a good generic picture but there are more possibilities.

9.1 Another way to find scalings: breakdown of ordering

Way back when we looked at regular expansions, I mentioned that one possible warning sign was that the ordering of terms in our expansion could break down. This can be used as an alternative method of seeking out new scalings and stretches, particularly for complex problems and when the outer scale and stretch are fixed by the boundary conditions.

This example comes from Hinch exercise 5.12 (and originally Van Dyke):

$$x^3 \frac{\mathrm{d}y}{\mathrm{d}x} = \varepsilon((1+\varepsilon)x + 2\varepsilon^2)y^2 \qquad \text{in } \ 0 < x < 1$$

with boundary condition $y(1) = 1 - \varepsilon$.

We start with the obvious expansion:

$$y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

and substitute to have

$$\begin{array}{rcll} x^3 y_0' & = & 0 & \\ \varepsilon x^3 y_1' & = & \varepsilon x y_0^2 & \\ \varepsilon^2 x^3 y_2' & = & 2\varepsilon^2 x y_0 y_1 & + \quad \varepsilon^2 x y_0^2 \end{array}$$

We will be applying the boundary condition to this solution: $y_0(1) = 1$, $y_1(1) = -1$, $y_2(1) = 0$, and so on.

Order 1 $y_0' = 0$ gives $y_0 = a_0$ and hence $y_0 = 1$.

Order ε $x^3 y_1' = x$ gives $y_1 = a_1 - 1/x$ and hence $y_1 = -1/x$.

Order ε^2 $x^3 y_2' = x - 2$ gives $y_2 = a_2 - 1/x + 1/x^2$ and hence $y_2 = 1/x^2 - 1/x$.

Our outer solution begins

$$y \sim 1 - \frac{\varepsilon}{x} + \varepsilon^2 \left(\frac{1}{x^2} - \frac{1}{x} \right) + \cdots$$

Now both the (nominally) order ε and order ε^2 terms become order 1 when $x \sim \varepsilon$. The function value is still order 1 (pick, for instance, $x = 2\varepsilon$ to see this) and so we look for an inner expansion with $x = \varepsilon z$ and put

$$y = f_0(z) + \varepsilon f_1(z) + \varepsilon^2 f_2(z) + \cdots$$

The differential equation transforms to

$$z^3 \frac{\mathrm{d}y}{\mathrm{d}z} = ((1+\varepsilon)z + 2\varepsilon)y^2$$

which then gives

$$\begin{array}{ccccc} z^3 f_0' & = & z f_0^2 & & \\ \varepsilon z^3 f_1' & = & 2\varepsilon z f_0 f_1 & + & \varepsilon(z+2) f_0^2 \\ \varepsilon^2 z^3 f_2' & = & \varepsilon^2 z (f_1^2 + 2 f_0 f_2) & + & 2\varepsilon^2 (z+2) f_0 f_1 \end{array}$$

We will solve for two terms before matching with the outer.

Order 1 $z^3 f_0' = z f_0^2$ gives $1/f_0 = A_0 + 1/z$ and $f_0 = z/(1 + A_0 z)$.

Order ε $z^3 f_1' - 2z f_0 f_1 = (z+2) f_0^2$ becomes

$$f_1' - 2f_1/z(1 + A_0 z) = (z+2)/z(1 + A_0 z)^2$$

and hence (using an integrating factor of $(1 + A_0 z)^2/z^2$) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{(1 + A_0 z)^2}{z^2} f_1 \right) = \frac{1}{z^2} + \frac{2}{z^3}$$

$$f_1 = \frac{A_1 z^2}{(1 + A_0 z)^2} - \frac{1 + z}{(1 + A_0 z)^2}.$$

So now our two solutions are

$$y_{\text{outer}} = 1 - \frac{\varepsilon}{x} + \varepsilon^2 \left(\frac{1}{x^2} - \frac{1}{x} \right) + \cdots$$

$$y_{\text{inner}} = \frac{z}{(1 + A_0 z)} + \varepsilon \left(\frac{A_1 z^2}{(1 + A_0 z)^2} - \frac{1 + z}{(1 + A_0 z)^2} \right) + \cdots$$

related by $x = \varepsilon z$. Introducing $x = \varepsilon^\alpha \eta$ and $z = \varepsilon^{\alpha-1}\eta$ and expanding (noting that z is large and so z^{-1} is small) gives

$$\begin{aligned}
y_{\text{outer}} &\sim 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{1}{\eta^2} - \varepsilon^{2-\alpha}\frac{1}{\eta} + \cdots \\
y_{\text{inner}} &= \frac{1}{A_0} - \frac{\varepsilon^{1-\alpha}}{A_0^2\eta} + \frac{\varepsilon^{2-2\alpha}}{\eta^2 A_0^3} + \cdots \\
&\quad + \varepsilon\left(\frac{A_1}{A_0^2(1+(A_0\varepsilon^{\alpha-1}\eta)^{-1})^2} - \frac{1+\varepsilon^{\alpha-1}\eta}{(1+A_0(\varepsilon^{\alpha-1}\eta))^2}\right) + \cdots
\end{aligned}$$

Clearly to match the order 1 term we need $A_0 = 1$; then the comparison becomes

$$\begin{aligned}
y_{\text{outer}} &\sim 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{1}{\eta^2} - \varepsilon^{2-\alpha}\frac{1}{\eta} + \cdots \\
y_{\text{inner}} &= 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{1}{\eta^2} - \varepsilon^{3-3\alpha}\frac{1}{\eta^3} + \cdots \\
&\quad + \varepsilon A_1(1+\varepsilon^{1-\alpha}\eta^{-1})^{-2} - \varepsilon^{2-\alpha}\eta^{-1}(1+\varepsilon^{1-\alpha}\eta^{-1})^{-1} + \cdots \\
&= 1 - \varepsilon^{1-\alpha}\frac{1}{\eta} + \varepsilon^{2-2\alpha}\frac{1}{\eta^2} - \varepsilon^{3-3\alpha}\frac{1}{\eta^3} + \cdots \\
&\quad + (A_1\varepsilon - 2A_1\varepsilon^{2-\alpha}\eta^{-1} + \cdots) - \varepsilon^{2-\alpha}\eta^{-1}(1 - \varepsilon^{1-\alpha}\eta^{-1} + \cdots)
\end{aligned}$$

There is nothing in the outer solution to match the $A_1\varepsilon$ term so we need $A_1 = 0$; the other unmatched terms all have powers like $\varepsilon^{3-n\alpha}$ so would match the third term of the outer, which we have not calculated. Our inner solution is therefore

$$y_{\text{inner}} = \frac{z}{(1+z)} - \frac{\varepsilon}{(1+z)} + \cdots$$

with $x = \varepsilon z$.

This problem has hidden depths though: the first two terms of our inner expansion break order when z is order ε. At that point the function value is also order ε, so we look for an inner-inner expansion $y = \varepsilon F(X)$ with $z = \varepsilon X$. The governing equation:

$$z^3\frac{\mathrm{d}y}{\mathrm{d}z} = ((1+\varepsilon)z + 2\varepsilon)y^2$$

becomes:

$$X^3\frac{\mathrm{d}F}{\mathrm{d}X} = ((1+\varepsilon)X + 2)F^2.$$

Here we will only look for the leading order term:

$$X^3\frac{\mathrm{d}F_0}{\mathrm{d}X} = (X+2)F_0^2 \qquad \int\frac{\mathrm{d}F_0}{F_0^2} = \int\frac{1}{X^2} + \frac{2}{X^3}\,\mathrm{d}X$$

$$\frac{-1}{F_0} = \frac{-1}{X} - \frac{1}{X^2} - C_0 \qquad F_0 = \frac{X^2}{C_0X^2 + X + 1}.$$

Now we need to compare the inner and double-inner expansions:

$$\begin{aligned}
y_{\text{inner}} &= \frac{z}{(1+z)} - \frac{\varepsilon}{(1+z)} + \cdots \\
y_{\text{double}} &\sim \varepsilon\frac{X^2}{C_0X^2 + X + 1}
\end{aligned}$$

with $z = \varepsilon X$. We set $z = \varepsilon^\alpha \xi$ and $X = \varepsilon^{\alpha-1}\xi$ to have

$$\begin{aligned}
y_{\text{inner}} &= \varepsilon^\alpha \xi(1 - \varepsilon^\alpha \xi + \cdots) - \varepsilon(1 + \cdots) \\
y_{\text{double}} &\sim \frac{\varepsilon}{C_0}(1 + \varepsilon^{1-\alpha} C_0^{-1} \xi^{-1} + \varepsilon^{2-2\alpha} C_0^{-1} \xi^{-2})^{-1} \\
&\sim \frac{\varepsilon}{C_0}(1 - \varepsilon^{1-\alpha} C_0^{-1} \xi^{-1} + \cdots)
\end{aligned}$$

The leading order terms here simply don't balance. There is nothing in the double-inner that gets as large as the $\varepsilon^\alpha \xi$ term in the inner. However, in expanding our double-inner solution, we did assume that C_0 was nonzero. If we try the case where it is zero, we get:

$$\begin{aligned}
y_{\text{double}} &= \varepsilon \frac{X^2}{X+1} + \cdots = \varepsilon X \frac{1}{1 + X^{-1}} + \cdots \\
&\sim \varepsilon^\alpha \xi(1 - \varepsilon^{1-\alpha} \xi^{-1} + \cdots)
\end{aligned}$$

which now matches the leading term from the inner. To match any more terms we would need to go to higher order in both expansions.

In summary, this ODE has three layers of asymptotic solution:

$$\begin{aligned}
y_{\text{outer}} &= 1 - \frac{\varepsilon}{x} + \varepsilon^2 \left(\frac{1}{x^2} - \frac{1}{x} \right) + \cdots \\
y_{\text{inner}} &= \frac{z}{(1+z)} - \frac{\varepsilon}{(1+z)} + \cdots \qquad \text{with } \ x = \varepsilon z \\
y_{\text{double}} &= \varepsilon \frac{X^2}{X+1} + \cdots \qquad \text{with } \ z = \varepsilon X.
\end{aligned}$$

This three-layered structure is known as a triple-deck problem.

9.2 A worse example

This example comes from the book by Cole. The governing equation is

$$\varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f \frac{\mathrm{d}f}{\mathrm{d}x} - f = 0$$

with boundary conditions $f(0) = -1$, $f(1) = 1$.

These boundary conditions fix f to be strictly order 1, so we cannot scale f and can only consider stretching x. Note that you have seen this equation before in exercise 4 of sheet 3. In the case of no scaling ($\alpha = 0$) you should have found two possible stretches: $x \sim 1$ and $x \sim \varepsilon$. You will also have found all the possible leading-order outer and inner solutions, but I didn't give you any boundary conditions and you hadn't learnt about matching yet, so you couldn't determine any of the constants.

Outer

Let us look first at the outer solution. We pose $f = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \cdots$. The leading-order equation is

$$f_0 \frac{\mathrm{d}f_0}{\mathrm{d}x} - f_0 = 0 \implies f_0 \left(\frac{\mathrm{d}f_0}{\mathrm{d}x} - 1 \right) = 0$$

which has two solutions, $f_0(x) \equiv 0$ and $f_0(x) = x + C$. Note that for both of these, $\mathrm{d}^2 f_0/\mathrm{d}x^2 = 0$ and so f_0 is an exact solution of the equation, and $f_1 = f_2 = \cdots = 0$.

Clearly the branch $f_0 = 0$ can't match either of the boundary conditions, so we know our outer solution must be

$$f(x) = x + C.$$

We have not yet found where the boundary layer will be; since the outer is so simple, we might as well work out the constant for both possibilities now.

If the outer meets $x = 1$ then we have $C = 0$ and so $f_{\text{outer},1}(x) = x$.
If the outer meets $x = 0$ then instead we have $C = -1$ and $f_{\text{outer},0}(x) = x - 1$.

Inner

What stretch do we expect for the inner? Note that the boundary conditions mean we can't scale f, we can only stretch x. We found in your exercise that we should stretch $x = a + \varepsilon z$.

We introduce $z = (x - a)/\varepsilon$ and rewrite our differential equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + f\frac{\mathrm{d}f}{\mathrm{d}z} - \varepsilon f = 0$$

Now we pose an inner expansion: $f \sim F_0(z) + \varepsilon F_1(z) + \varepsilon^2 F_2(z) + \cdots$, and at leading order the governing equation is

$$\frac{\mathrm{d}^2 F_0}{\mathrm{d}z^2} + F_0\frac{\mathrm{d}F_0}{\mathrm{d}z} = 0.$$

We can integrate this directly once:

$$\frac{\mathrm{d}F_0}{\mathrm{d}z} + \frac{1}{2}F_0^2 = C.$$

Now remember that for a boundary layer solution, we are going to need solutions which *decay* to some fixed value out of the layer. This means that as $z \to \pm\infty$ (but not necessarily both), we need $\mathrm{d}F_0/\mathrm{d}z \to 0$ and so $C \geq 0$. (This already eliminates some of the possible solutions you found.) Let us set $C = 2k^2$ for convenience.

This ODE for F_0 has three different possible forms of solution. If $k = 0$ the solution is

$$F_0 = \frac{2}{z + D},$$

with arbitrary constant D. If we are to use this solution the point $z = -D$ must not lie within our domain.

For $k > 0$ there are three solutions, two of which work.

First we look at the possibility that $|F_0| = 2k$. In that case

$$\frac{\mathrm{d}F_0}{\mathrm{d}z} = 0 \qquad F_0 = \pm 2k.$$

This is not a true inner solution: it does not depend on z, so it doesn't vary quickly w.r.t. x. In fact, it is just a regular outer solution expanded in terms of the inner variable. So we move on to the two other cases: $|F_0| < 2k$ and $|F_0| > 2k$.

In both of these cases we can solve the ODE by partial fractions:

$$2\frac{\mathrm{d}F_0}{\mathrm{d}z} = 4k^2 - F_0^2.$$

$$\int 2k \ \mathrm{d}z = \int \frac{4k}{4k^2 - F_0^2}\,\mathrm{d}F_0 = \int \left(\frac{1}{2k - F_0} + \frac{1}{2k + F_0}\right)\mathrm{d}F_0$$

$$2kz + 2B = -\ln|2k - F_0| + \ln|2k + F_0| = \ln\left|\frac{2k + F_0}{2k - F_0}\right|$$

$$\frac{2k + F_0}{2k - F_0} = \pm\exp\left[2(kz + B)\right]$$

$$F_0 = 2k\frac{\exp\left[(kz + B)\right] \mp \exp\left[-(kz + B)\right]}{\exp\left[(kz + B)\right] \pm \exp\left[-(kz + B)\right]}$$

which has two solutions,

$$F_0 = 2k\tanh\left[(kz + B)\right] \qquad F_0 = 2k\coth\left[(kz + B)\right],$$

both of which decay exponentially to some limit as $z \to \infty$.

Look at the forms of the tanh and coth curves:

We can see that the tanh solution moves smoothly from one value to another over the width of the boundary layer, whereas the coth profile cannot be given a value $z = 0$. This means that the coth profile can only be used if the boundary layer is at one end or other of the region, whereas the tanh profile can be used anywhere.

Matching with a single boundary layer

Let us try first to put the boundary layer near $x = 0$. The outer solution must match the boundary condition at $x = 1$ so

$$f_{\text{outer}} = x.$$

Now in the inner region we can either have

$$F(z) = 2k\tanh\left[kz + B\right] \quad \text{or} \quad F(z) = 2k\coth\left[kz + B\right] \quad \text{or} \quad F(z) = 2/(z + D).$$

In each case we need $F(z = 0) = -1$ and $F(z \to \infty) = 0$. The second of these gives $k = 0$ both the first two cases, and then we cannot match the other boundary condition for any B. For the third function, we have the right result as $z \to \infty$, but to match the condition at $z = 0$ gives $D = -2$ and the forbidden point $z = -D = 2$ lies within our domain. **FAILED.**

Now we try with a boundary layer near $x = 1$. This time the outer solution must match the boundary condition at $x = 0$ so

$$f_{\text{outer}} = x - 1.$$

In the inner region the possibilities are

$$F(z) = 2k \tanh[kz + B] \quad \text{or} \quad F(z) = 2k \coth[kz + B] \quad \text{or} \quad F(z) = 2/(z + D).$$

The boundary conditions are $F(z = 0) = 1$ and $F(z \to -\infty) = 0$. We have the same problem again: we need both $k \neq 0$ and $k = 0$, or $z = -D$ lies within our domain. **FAILED.**

Finally, let us try having the "boundary layer" in the middle, at some general position a between 0 and 1. This time we have two different branches of the outer solution:

$$f_{\text{outer},1}(x) = x \qquad f_{\text{outer},1}(a) = a.$$
$$f_{\text{outer},0}(x) = x - 1 \qquad f_{\text{outer},0}(a) = a - 1.$$

Our inner solution will then have boundary conditions

$$F(z \to -\infty) = a - 1 \qquad F(z \to \infty) = a.$$

The only profile we are allowed is the tanh profile, which goes from $-2k$ to $2k$ over the width of the layer. This fixes

$$a - 1 = -2k \qquad a = 2k \qquad \Longrightarrow a = 1/2, \; k = 1/4.$$

Our leading-order inner solution is

$$F(z) = \frac{1}{2} \tanh[z/4]$$

and $z = (x - \frac{1}{2})/\varepsilon$. The complete solution looks like this:

Note: It is also possible to construct a solution having more than one boundary layer: for example, try putting a tanh boundary layer at each end. However, a single localised region of "failure" is more physically realistic.

Further expansion

Since the solution we have found in the inner is not an exact solution, we could continue to higher orders. Often you will find that the later equations are easier to solve than the first because the new terms come in linearly. Although the equation becomes linear, it's not really easier in this case; but let us try calculate one more term. Recall we had

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + f \frac{\mathrm{d}f}{\mathrm{d}z} - \varepsilon f = 0$$

with

$$f \sim \frac{1}{2} \tanh[z/4] + \varepsilon F_1(z) + \cdots$$

At order ε this gives

$$\frac{\mathrm{d}^2 F_1}{\mathrm{d}z^2} + F_0\frac{\mathrm{d}F_1}{\mathrm{d}z} + F_1\frac{\mathrm{d}F_0}{\mathrm{d}z} - F_0 = 0$$

$$\frac{\mathrm{d}^2 F_1}{\mathrm{d}z^2} + \frac{1}{2}\tanh\left[\frac{z}{4}\right]\frac{\mathrm{d}F_1}{\mathrm{d}z} + \frac{1}{8}\mathrm{sech}^2\left[\frac{z}{4}\right]F_1 = \frac{1}{2}\tanh\left[\frac{z}{4}\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{\mathrm{d}F_1}{\mathrm{d}z} + \frac{1}{2}\tanh\left[\frac{z}{4}\right]F_1\right\} = \frac{1}{2}\tanh\left[\frac{z}{4}\right]$$

$$\frac{\mathrm{d}F_1}{\mathrm{d}z} + \frac{1}{2}\tanh\left[\frac{z}{4}\right]F_1 = 2\ln\cosh\left[\frac{z}{4}\right] + C_1$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\cosh^2\left[\frac{z}{4}\right]F_1\right\} = 2\cosh^2\left[\frac{z}{4}\right]\ln\cosh\left[\frac{z}{4}\right] + C_1\cosh^2\left[\frac{z}{4}\right]$$

which may be integrated to give the solution:

$$\begin{aligned} F_1 = C_1\left(\frac{z}{4} + \sinh\left[\frac{z}{4}\right]\right)\mathrm{sech}^2\left[\frac{z}{4}\right] + C_2\mathrm{sech}^2\left[\frac{z}{4}\right] \\ + 2\mathrm{sech}^2\left[\frac{z}{4}\right]\int \cosh^2\left[\frac{z}{4}\right]\ln\cosh\left[\frac{z}{4}\right]\,\mathrm{d}z \end{aligned}$$

Unfortunately the final integral is only available in terms of the polylogarithm function:

$$\begin{aligned} \int \cosh^2\left[\frac{z}{4}\right]\ln\cosh\left[\frac{z}{4}\right]\,\mathrm{d}z = \frac{z}{4} - \frac{z^2}{16} - \frac{1}{2}\sinh\left[\frac{z}{2}\right] \\ - \frac{z}{2}\ln\left(1 + \exp\left[-\frac{z}{2}\right]\right) + \mathrm{Li}_2\left(-e^{-z/2}\right) + \ln\cosh\left[\frac{z}{4}\right]\left(\frac{z}{2} + \sinh\left[\frac{z}{2}\right]\right) \end{aligned}$$

in which

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^2},$$

but if we had been able to find F_1 in terms of more useful functions, the matching procedure would have continued as before.

Analytical Methods: Exercises 1

1. Try a regular perturbation expansion in the following differential equation:
$$y'' + 2\varepsilon y' + (1+\varepsilon^2)y = 1, \qquad y(0) = 0, \qquad y(\pi/2) = 0.$$
Calculate the first three terms, that is, up to order ε^2. Apply the boundary conditions at each order.

2. Calculate the first two nonzero terms of a regular expansion in ε for the following integral:
$$I = \int_0^\varepsilon \frac{\mathrm{d}x}{(\varepsilon^2 - x^2 + \cos\varepsilon - \cos x)^{1/2}}.$$
[Hint: you will need to keep terms of order ε^4 initially.]

3. Find the general solution to the PDE for $f(\theta,\phi)$:
$$\frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta f) + \frac{1}{a\sin\theta}\frac{\partial}{\partial\phi}(v_\phi f) + \sin^2\theta\cos 2\phi = 0$$
in which
$$\begin{aligned} v_\theta &= a\sin\theta\cos\theta\cos 2\phi \\ v_\phi &= -a\sin\theta\sin 2\phi \end{aligned}$$

4. Consider the problem
$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = 0$$
in $x \geq 0$, $t \geq 0$, with initial and boundary conditions
$$u(x,0) = \sqrt{x} \qquad u(0,t) = 0.$$
Find the general solution implicitly and hence the specific solution in this case.

5. How would you expect the diameter of a spider's web to scale with L, the bodylength of the spider? Be clear about any assumptions you make.

Answers

1. $y = 1 - \cos x - \sin x + \varepsilon[(x-\pi/2)\sin x + x\cos x]$
$\quad - \varepsilon^2[1 + (x^2/2 - \pi x/2 - 1 + \pi^2/8)\sin x + (x^2/2 - 1)\cos x]$.

2. $I = \dfrac{\pi}{\sqrt{2}}\left(1 - \dfrac{\varepsilon^2}{16} + O(\varepsilon^4)\right)$.

3. $f(\theta,\phi) = F(\sin 2\phi\tan^2\theta)\sec^3\theta + \dfrac{1}{3}$.

4. Implicit solution $u = F(x - u^2 t)$, particular solution $u(x,t) = \sqrt{\dfrac{x}{(1+t)}}$.

5. If we assume that two quantities: the thickness of the spun fibre, and the hole-size in the finished web, are both independent of spider size, then the diameter scales as $L^{3/2}$.

Analytical Methods: Exercises 2

1. Try a dilation transformation on the Burger's equation: $u_t + uu_x = 0$. Find the specific solution for initial conditions
$$u(x,1) = \frac{x + (x^2-1)^{1/2}}{2}$$
and show it matches that obtained by the method of characteristics.

2. Find the distinguished scalings, and the first two terms in the expansion of each root, for the following equation:
$$\varepsilon x^3 + x^2 + (2-\varepsilon)x + 1 = 0.$$

3. Find the first two terms of all four roots of $\varepsilon x^4 - x^2 - x + 2 = 0$.

4. Work out the first two terms in an expansion of each solution to $xe^{-x} = \varepsilon$.

5. Verify that the function
$$u = \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x',t')\,\mathrm{d}x'\,\mathrm{d}t'$$
satisfies the inhomogeneous wave equation $u_{tt} - c^2 u_{xx} = F(x,t)$.

6. [Weinberger p.40] Find the characteristics through $(0,1)$ for the equation
$$\frac{\partial^2 u}{\partial t^2} - e^{2x}\frac{\partial^2 u}{\partial x^2} = 0.$$

7. Find two terms of a regular perturbation expansion for $f(x,t)$ in:
$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \varepsilon \cos x f = x$$
with boundary conditions $f(x,0) = \partial f/\partial t(x,0) = 0$. This particular problem can be solved in the same way even if $\varepsilon = 1$: this is the method of *successive approximations*. [Ref: Weinberger p. 384.]

Answers

1. $u = t^{m-1}f(\xi)$ with $\xi = t^{-m}x$ and $(f(\xi) - m\xi)f'(\xi) + (m-1)f(\xi) = 0$. Specific solution $u = (x/t + [(x/t)^2 - t^{-1}]^{1/2})/2$.

2. Scalings $x \sim 1$ and $x \sim \varepsilon^{-1}$; roots $x = -1-2\varepsilon+O(\varepsilon^2)$, $x = -\varepsilon^{-1}+2+O(\varepsilon)$, $x = -1$ (exact solution, no further terms).

3. $x \sim 1 + \varepsilon/3$; $x \sim -2 - 16\varepsilon/3$; $x \sim \varepsilon^{-1/2} + 1/2$; $x \sim -\varepsilon^{-1/2} + 1/2$.

4. $xe^{-x} = \varepsilon$. There are two roots: $x \sim \varepsilon + \varepsilon^2$ and $x \sim \ln(1/\varepsilon) - \ln(\ln(1/\varepsilon))$.

6. $t = e^x$ and $t = 2 - e^x$.

7. $f(x,t) = \frac{1}{2}xt^2 + \varepsilon\left[\frac{1}{2}t^2(x\cos x - 2\sin x) - x\cos x + 4\sin x\right.$
$\left. + \frac{1}{2}(x+t)\cos(x+t) + \frac{1}{2}(x-t)\cos(x-t) - 2\sin(x+t) - 2\sin(x-t)\right]$

Analytical Methods: Exercises 3

1. Find where the following operators are hyperbolic, parabolic, and elliptic:

$$\text{(a)}\quad \frac{\partial^2 u}{\partial t^2} + t\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} \qquad \text{(b)}\quad t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.$$

2. Solve the following PDE with the boundary conditions given:

$$\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{(t+1)^2}\frac{\partial^2 u}{\partial x^2} = 0 \qquad u(x,0) = u(1,t) = u(2,t) = 0.$$

3. Find the distinguished stretches, and the leading term of each solution:

$$\varepsilon^3\frac{\mathrm{d}^3 f}{\mathrm{d}x^3} + \varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} + f = 0.$$

4. $\varepsilon\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\dfrac{\mathrm{d}f}{\mathrm{d}x} - f = 0.$
   (a) Find the scalings $f = \varepsilon^\alpha F$ and stretches $x = a + \varepsilon^\beta z$ at which two dominant terms balance, and sketch these scalings in the α–β plane.
   (b) Hence determine the critical α and β for all three terms to balance.
   (c) Give also the possible values of β if the boundary conditions fix $\alpha = 0$. Find the leading term in an expansion for f in each case.

5. Consider the following equation and boundary conditions:

$$\varepsilon\frac{\partial^2 u}{\partial x^2} + \varepsilon\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0$$

$$u(-1,y) = u(1,y) = 0 \qquad u(x,1) = 1 - x^2 \qquad \varepsilon\frac{\partial u}{\partial y}(x,0) + u(x,0) = 0.$$

   (a) Calculate the first two terms of a regular perturbation expansion, ignoring the boundary condition at $y = 0$. Satisfy the other three boundary conditions at leading order only.
   (b) What scaling can be applied to y to find another solution? Calculate two terms of this solution, using only the $y = 0$ boundary condition.
   (c) Taking ε as a normal parameter (i.e. forgetting that it is small), find the full solution to the problem by separating variables. You need not determine all the coefficients in the sum; but find the general solution satisfying the x-boundary conditions.
   (d) Comment on the structure of your general solution when ε is small.

Answers [Note: questions 1 & 2 are from Weinberger]

1. (a) H $t^2 > 4x$; P $t^2 = 4x$; E $t^2 < 4x$. (b) H $xt < 1$; P $xt = 1$; E $xt > 1$.

2. $u(x,t) = \displaystyle\sum_n \alpha_n x^{1/2}(t+1)^{1/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right) \sin\left(\frac{n\pi \ln(t+1)}{\ln 2}\right).$

3. $1,\ \varepsilon,\ \varepsilon^2$. $f = be^{-x}$; $f = be^{-(x-a)/\varepsilon} + c$; $f = be^{-(x-a)/\varepsilon^2} + c(x-a)/\varepsilon^2 + d$.

4. (a) $\alpha + \beta = 1$, $\alpha < \beta$; $\beta = 1/2$, $\alpha > \beta$; $\alpha = \beta$, $\beta < 1/2$. (b) $\alpha = \beta = 1/2$.
   (c) $\beta = 0$: $f_0 = a + x$. $\beta = 1$: any of $F_0 =$ constant, $F_0 = 2(z+b)^{-1}$, $F_0 = -2k\tan[k(z+b)]$, $2k\tanh[k(z+b)]$, or $2k\coth[k(z+b)]$.

5. (a) $u \sim 1 - x^2 + \varepsilon(2y + f_1(x)) + \cdots$
   (b) The scaling is $y = a + \varepsilon Y$ giving $f \sim A_0(x)e^{-Y} + \varepsilon A_1(x)e^{-Y} + \cdots$.
   (c) $u = \displaystyle\sum_n a_n \cos\left[\frac{(2n+1)\pi x}{2}\right](c_n \exp[m_1 y] + d_n \exp[m_2 y])$ with $m_1,\ m_2 = [-1 \pm \sqrt{1 + (2n+1)^2\pi^2\varepsilon^2}]/2\varepsilon$.

Analytical Methods: Exercises 4

1. Look at the problem
$$\varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} = \cos x$$
with boundary conditions $f(0) = 0$, $f(\pi) = 1$. Find the two distinguished stretches for this equation. Calculate the first three terms of the regular expansion, and apply the boundary condition at π to determine the constants.

   Now apply your stretch near $x = 0$. Find the first three terms of the inner solution, and apply the boundary condition at $x = 0$ to determine some of the constants in this expansion.

   Finally use an intermediate variable to match your two expressions and determine the remaining constants.

2. Calculate three terms of the outer solution of
$$(1+\varepsilon)x^2 y' = \varepsilon((1-\varepsilon)xy^2 - (1+\varepsilon)x + y^3 + 2\varepsilon y^2) \quad \text{in } \ 0 < x < 1$$
with $y(1) = 1$. Locate the non-uniformity of the asymptoticness, and hence the rescaling for an inner region. Thence find two terms for this inner solution.

3. Find the image of the unit disc $|z-1| \leq 1$ under the mapping $w = 1/z$.

4. Find the image of $-\pi/2 < x < \pi/2$, $0 < y < 1$ under $w = \sin z$.

5. Find the image of $-\pi/4 < x < \pi/4$, $-1 < y < 1$ under $w = \sin z$.

6. Solve the problem $\nabla^2 u = 0$ for $1 < r < e^{\alpha}$, $0 < \alpha < \pi$ with boundary conditions
$$\partial u/\partial r(1,\theta) = 0 \qquad \partial u/\partial r(e^{\alpha},\theta) = \sin\theta \qquad u(r,0) = 0 \qquad u(r,\pi) = 0$$
   (a) by separation of variables, and
   (b) using the transformation $w = \ln z$.

Answers [Note: question 2 is from Hinch; 3–6 are from Weinberger]

1. $\delta = 1$, $\delta = \varepsilon$. Outer: $f = 1 + \sin x - \varepsilon[1+\cos x] - \varepsilon^2 \sin x + \cdots$
Inner ($x = \varepsilon z$): $f = a_0 - a_0 e^{-z} + \varepsilon[a_1 - a_1 e^{-z} + z] + \varepsilon^2[a_2 - a_2 e^{-z}] + \cdots$
After matching: $f(z) = 1 - e^{-z} + \varepsilon[2e^{-z} - 2 + z] + O(\varepsilon^3)$.

2. Outer $y \sim 1 + \varepsilon[1 - 1/x] + \varepsilon^2[1/2 - 2/x + 3/(2x^2)] + \cdots$.
Inner (with $x = \varepsilon z$): $y \sim (1+2/z)^{-1/2} + \varepsilon[(1+1/z)(1+2/z)^{-3/2}] + \cdots$

3. $\mathrm{Real}\,(w) \geq 1/2$.

4. Putting $w = \eta + i\xi$, the image is $(\eta/\cosh 1)^2 + (\xi/\sinh 1)^2 \leq 1$, $\xi \geq 0$.

5. Putting $w = \eta + i\xi$, the image is the curvilinear rectangle bounded by the hyperbola $\eta^2 - \xi^2 = 1/2$ and the ellipse $(\eta/\cosh 1)^2 + (\xi/\sinh 1)^2 = 1$.

6. $u(r,\theta) = (r + r^{-1})\sin\theta/(1 - e^{-2\alpha})$.

Analytical Methods: Exercises 5

1. [Blasius boundary layer] Consider the steady Navier-Stokes equations:

$$\underline{\nabla} \cdot \underline{u} = 0 \qquad \underline{u} \cdot \underline{\nabla u} = -\underline{\nabla} p + \nabla^2 \underline{u},$$

   here made dimensionless using a typical velocity U, the fluid density ρ and the viscous lengthscale $L = \eta/\rho U$.

   Investigate flow past a semi-infinite flat plate (no natural lengthscale):

$$\underline{u} \to \underline{e}_x \ \text{ at } \infty; \qquad \underline{u} = 0 \ \text{ on } x \geq 0,\ y = 0.$$

   A dilation transformation is appropriate; expect the vertical velocity to be a smaller scale than the horizontal. You may find it easiest to work with a streamfunction $u = \partial\psi/\partial y$. The ODE which results can only be solved numerically.

2. Find the asymptotic behaviour of

$$J_\nu(\nu z) = \frac{1}{2\pi i} \int_{\infty - i\pi}^{\infty + i\pi} \exp\left[\nu z \sinh t - \nu t\right] \,\mathrm{d}t$$

   for fixed real z with $0 < z < 1$ as $\nu \to \infty$.

Answers

1. The leading-order scalings are $u = U(\xi)$, $v = x^{-1/2}V(\xi)$ and $p = P_0$ (a constant), in which $\xi = x^{-1/2}y$. A streamfunction gives

$$\psi = x^{1/2} f(\xi) \ \text{ with } \ U(\xi) = f'(\xi), \quad V(\xi) = \frac{1}{2}[\xi f'(\xi) - f(\xi)].$$

   and the resulting ODE is $2f'''(\xi) + f(\xi)f''(\xi) = 0$, with $f(0) = f'(0) = 0$ and $f'(\xi) \to 1$ as $\xi \to \infty$.

2. $J_\nu(\nu z) \sim \left(\dfrac{1}{2\pi\nu(1-z^2)^{1/2}}\right)^{1/2} \exp\left[\nu\left((1-z^2)^{1/2} - \operatorname{arccosh}(1/z)\right)\right].$