CHAPTER II

The Lie algebra

In this chapter, we introduce the Lie algebra $\mathfrak{g}$ of a Lie group G and disuss its basic properties. We prove the fundamental principle that a (Lie-group) representation of G is determined 'infinitesimally', that is, by the associated representation of $\mathfrak{g}$.

We then discuss one-parameter subgroups and the exponential map and state and sketch the proof of the 'closed subgroup theorem' (Theorem II.2.13). We show that the exponential map for abelian Lie groups is a homomorphism and deduce that every connected (finite-dimensional) abelian Lie group has the form $\mathbb{T}^n \times \mathbb{R}^m$. This is an important result in the development of the representation theory of compact Lie groups, where restriction of a representation to the maximal torus is a key step.

1. Definition of Lie algebra

Definition II.1.1. A (finite-dimensional) Lie algebra $(V, [\cdot,\cdot])$ consists of a vector space V and a bilinear map (the Lie bracket)

$$(v, v') \mapsto [v, v'] : V \times V \longrightarrow V$$

which is skew-symmetric and satisfies the Jacobi identity:

$$[u, v] = -[v, u], \quad [u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \tag{II.1.1}$$

for all $u, v, w \in V$. A Lie algebra is real or complex according as V is a vector space over $\mathbb{R}$ or $\mathbb{C}$. V is *abelian* if $[v, w] = 0$ for all $v, w \in V$.

We shall see in the next two sections that if G is a Lie group then T_eG (the tangent space at the identity) is in a natural way a Lie algebra. This is usually denoted $\mathfrak{g}$.

Let us give some examples:

Example II.1.2. Any vector space V with the $[\cdot,\cdot]$ identically zero is a Lie algebra. We call such Lie algebras *abelian*. We shall see that in the correspondence between groups and Lie algebras, the Lie group is abelian iff the Lie algebra is abelian.

Example II.1.3. Let $K = \mathbb{R}$ or $\mathbb{C}$. We write $\mathfrak{gl}_n(K) = \mathrm{End}(K^n)$ with matrix commutator as Lie bracket. We'll see that $\mathfrak{gl}_n(K)$ is the Lie algebra of $\mathrm{GL}_n(K)$.

Example II.1.4. An infinite-dimensional example: the space $\mathrm{Vect}(M)$ of smooth vector fields on a (finite-dimensional) manifold M, with commutator of vector fields as Lie bracket.

Example II.1.5. It follows that for any Lie group G that is a subgroup of GL_n, the Lie algebra $\mathfrak{g}$ of G is a linear subspace of $\mathfrak{gl}_n$ and the Lie bracket is given by commutator of matrices.

The Lie algebra $\mathfrak{u}_n$, for example, consists of the skew-adjoint elements of $\mathfrak{gl}_n(\mathbb{C})$. If you have never done so, check that commutator preserves skew-adjointness, that is

$$X, Y \in \mathfrak{u}_n \Rightarrow [X, Y] \in \mathfrak{u}_n \,.$$

Note however that XY is not generally in $\mathfrak{u}_n$. Ordinary product is not part of the furniture in a Lie algebra.

Observe that for $\mathfrak{gl}_n(\mathbb{R})$, and hence for its subalgebras, the two properties in (II.1.1) are satisfied. (Simple computation.)

Definition II.1.6. A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ which is closed under bracket, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. A homomorphism of Lie algebras $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a linear map which preserves Lie bracket:

$$\varphi[X, Y] = [\varphi(X), \varphi(Y)] \text{ for all } X, Y \in \mathfrak{g}\,.$$

Here of course the bracket on the LHS is that of $\mathfrak{g}$ and the bracket on the RHS is that of $\mathfrak{g}'$.

A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}_N$. (Real if $\mathfrak{g}$ is real and $\mathfrak{gl}_N$ is real, complex if $\mathfrak{g}$ and $\mathfrak{gl}_N$ are complex.)

Thus we have a *category* of Lie algebras.

1.1. Lie-algebra structure of T_eG—first definition. Let G be a (finite-dimensional) Lie group. Like any group, G acts on itself in three ways,

- Left action $(g, x) \mapsto gx$;
- Right action $(g, x) \mapsto xg^{-1}$;
- Conjugation action $(g, x) \mapsto gxg^{-1}$.

In each case although g and x are both elements of G, g is the one that is acting so that we have the rule $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ where $\cdot$ denotes any of the above actions.

A Lie group is also a smooth manifold. The smooth structure gives us the option to differentiate, thus reducing (in principle) to linear algebra.

Our first definition arises by differentiating the conjugation action of G on itself. Denote the conjugation action Ψ, so $\Psi_g(x) = gxg^{-1}$. Then for each g, Ψ_g is a diffeomorphism of G and $\Psi_g(e) = e$. For each $g \in G$, therefore, the derivative $\mathrm{d}\Psi_g : T_eG \to T_eG$ is a linear map. This is the *adjoint action* of the group on T_eG:

$$\mathrm{d}\Psi_g = \mathrm{Ad}_g : T_eG \to T_eG. \tag{II.1.2}$$

The map $g \mapsto \mathrm{Ad}_g$ is a Lie group representation $G \to \mathrm{Aut}(T_eG)$, the *adjoint representation.* We have extracted a canonical representation of G in this way, but it is still nonlinear in G. But we can linearize this representation with respect to g, at $g = e$, and this gives adjoint action of T_eG on itself. More precisely, denote by $\mathrm{ad}(X)$ the derivative $\mathrm{d}\,\mathrm{Ad}$ in the direction X, at the point e. Then

$$\mathrm{ad} : T_eG \longrightarrow \mathrm{End}(T_eG). \tag{II.1.3}$$

Theorem II.1.7. *The tangent space T_eG, equipped with the bilinear operation*

$$[X, Y] = \mathrm{ad}(X)Y$$

is a Lie algebra.

This is *the* Lie algebra of G, usually denoted $\mathfrak{g}$.

PROPOSITION II.1.8. *The Lie algebra of* $\mathrm{GL}_n(K)$*, where* $K = \mathbb{R}$ *or* $\mathbb{C}$*, is* $\mathrm{End}(K^n)$*, with* $\mathrm{ad}(X)Y = [X, Y]$ *(matrix commutator).*

PROOF. It is clear that the tangent space at the identity is $\mathrm{End}(K)$ as claimed. To calculate ad, let $y = 1 + Y$, $Y \in \mathrm{End}(K)$. Then clearly

$$\mathrm{Ad}_g(Y) = gYg^{-1}, \quad Y \in \mathrm{End}(K).$$

Now we must differentiate with respect to g. If $g = 1 + X$, then

$$g^{-1} = 1 - X + O(X^2)$$

and so

$$\mathrm{Ad}_g(Y) = (1 + X)Y(1 - X + O(X^2) = 1 + XY - YX + O(X^2).$$

The part of this linear in X is thus the commutator

$$[X, Y] = XY - YX.$$

□

It is at least highly plausible that the Lie algebras of closed subgroups of $\mathrm{GL}_n(K)$ are then given by the corresponding subalgebras of matrices, the bracket operation still being matrix commutator.

To prove Theorem II.1.7 in generality, we need to show that $\mathrm{ad}(X)Y$ is skew in X and Y and satisfies the Jacobi identity. We shall at least sketch out why this is the case. Full details can be found in Chapter 2 of Adams.

1.2. Generalization. Suppose that $R : G \to H$ is a Lie group homomorphism. Denoting by Ψ the conjugation action on either of G or H, the fact that R is a homomorphism means

$$R \circ \Psi_g = \Psi_{R(g)} \text{ for all } g \in G. \tag{II.1.4}$$

We can now play the same game as we did previously (which is really the case $R = \mathrm{Id}$. That is, we differentiate either side of (II.1.4) at the identity, yielding

$$\rho \circ \mathrm{Ad}_g = \mathrm{Ad}_{R(g)} \circ \rho \tag{II.1.5}$$

as maps $T_eG \to T_eH$. Here, for readability, we've written $\rho = \mathrm{d}R_e$. Differentiating now with respect to g, we obtain Differentiating now with respect to g,

$$\rho(\mathrm{ad}(X)Y) = \mathrm{ad}(\rho(X))\rho(Y). \tag{II.1.6}$$

This shows that $\rho = \mathrm{d}R_e$ is a homomomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{h}$, apart from not actually having yet proved that $\mathrm{ad}(X)Y$ is a Lie bracket!

> The derivative ρ at e of a homomorphism of Lie groups $R : G \to H$ is a homomorphism of Lie algebras $\rho : \mathfrak{g} \longrightarrow \mathfrak{h}$.

This is referred to by Fulton–Harris as the 'First Principle'. In particular, for the case $H = \mathrm{GL}_N$, $\mathfrak{h} = \mathfrak{gl}_N$, it shows that a *representation* of G induces a representation of $\mathfrak{g}$. Under mild topological assumptions, there is a converse:

> Let G and H be Lie groups, with G simply connected. Then every lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{h}$ arises as the derivative at e of a Lie-group homomorphism $R : G \to H$.

We will probably not prove the second of these principles, but it is of course a crucial simplification to have in mind, because finding representations of Lie algebras is a much simpler problem.

1.3. Proof of Theorem II.1.7. We show first that $\mathrm{ad}(X)X = 0$ for all X. This implies the skew symmetry of $\mathrm{ad}(X)Y$ by polarization. Given $X \in T_eG$, there is a one-parameter subgroup $\varphi : \mathbb{R} \to G$, with $\dot{\varphi}(0) = X$. (See Proposition II.2.4 below.) We have

$$\varphi(t)\varphi(s)\varphi(t)^{-1} = \varphi(t)\varphi(s)\varphi(-t) = \varphi(s)$$

for all reall s, t. Differentiating this with respect to s and setting $s = 0$ gives

$$\varphi(t)X\varphi(-t) = X \text{ for all } t.$$

Further differentiating with respect to t and setting $t = 0$ gives $\mathrm{ad}(X)X$ on the LHS and 0 on the RHS.

Now we verify the Jacobi identity. We do this by pure thought as follows. Let $R : G \longrightarrow \mathrm{Aut}(T_eG)$ be the Adjoint representation, and let ρ be its derivative at e. Writing out the naturality (II.1.6) explicitly,

$$\rho[X,Y]Z = [\rho(X), \rho(Y)]Z.$$

But $\rho(A)B = [A, B]$, so

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]$$

which is equivalent to the Jacobi identity.

2. One-parameter subgroups and the exponential map

Definition II.2.1. A *one-parameter subgroup* of a Lie group G is a Lie group homomorphism

$$\varphi : \mathbb{R} \longrightarrow G \tag{II.2.1}$$

We often use the abbreviation 1PS for a one-parameter subgroup.

Definition II.2.2. A vector field v on G is left-invariant if

$$g_*(v_e) = v_g \text{ for all } g \in G \tag{II.2.2}$$

where here g just stands for the left-translation diffeomorphism $G \to G$ and g_* is the map of tangent spaces

$$T_eG \longrightarrow T_gG.$$

These two definitions are linked by the following simple result:

Proposition II.2.3. *Suppose that $\varphi : \mathbb{R} \to G$ is a 1PS. Then*

$$\dot{\varphi} : T_t\mathbb{R} = \mathbb{R} \longrightarrow T_{\varphi(t)}G \tag{II.2.3}$$

is left-invariant (along the image of φ in G). Conversely, if $\gamma : \mathbb{R} \to G$ is any curve with $\gamma(0) = e$ and $\dot{\gamma}$ is left-invariant along γ, then γ is a 1PS.

PROOF. Observe that if $\varphi : \mathbb{R} \to G$ is a 1PS, then $\dot{\varphi}(t) : \mathbb{R} \to T_{\varphi(t)}(G)$ is a linear map. Moreover,

$$\varphi(t+h) = \varphi(t)\varphi(h) \text{ so } \varphi(t)^{-1}\varphi(t+h) = \varphi(h)$$

It follows that $\varphi(t)^{-1}\dot{\varphi}(t) = \dot{\varphi}(0)$, in other words $\dot{\varphi}(t)$ is the left-invariant vector field through $\dot{\varphi}(0)$. $\square$

PROPOSITION II.2.4. *Given any $X \in T_eG$, there exists a unique 1PS φ_X such that $\dot{\varphi}_X(0) = X$.*

PROOF. φ_X is defined, in local coordinates, by a nonlinear ODE (the flow of the left-invariant vector field equal to X at e). By standard ODE theory, this can be solved near e, and then the local solution can be extended using the group law. See Adams, Theorem 2.6, p. 8 for full details. $\square$

DEFINITION II.2.5. The exponential map $\exp : \mathfrak{g} \to G$ is defined by $X \mapsto \varphi_X(1)$, where φ_X is the 1PS with derivative at 0 equal to X.

THEOREM II.2.6. $\exp$ *is smooth and* $d\exp : T_eG \to T_eG$ *is the identity map.*

PROOF. For the smoothness, we need to know that $X \mapsto \varphi_X(1)$ depends smoothly upon $X \in T_eG$. Now $\varphi_X(t)$ is the solution of an ODE with smooth coefficients which depend smoothly on X. Standard results from the theory of ODE give that $\varphi_X(t)$ depends smoothly on (X, t), and in particular $\varphi_X(1)$ depends smoothly on X. The statement about the derivative follows from the definition. For more details, see Adams, p.11. $\square$

It follows from the inverse function theorem that exp is a local diffeomorphism from a neighbourhood of $0 \in \mathfrak{g}$ to a neighbourhood of $e \in G$. Moreover exp is a natural map: given $R : G \to H$ a Lie group homomorphism with derivative $\rho : \mathfrak{g} \to \mathfrak{h}$, there is a commutative diagram

$$\begin{tikzcd} G \arrow[r, "R"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\rho"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

where $\rho = \mathrm{d}R_e$.

EXAMPLE II.2.7. For GL_n, the exponential map is the matrix exponential defined by the usual power series,

$$\exp(X) = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{II.2.4}$$

EXAMPLE II.2.8. For $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, $\mathfrak{t} = \mathbb{R}^n$ and the exponential map is $(t_1, \ldots, t_n) \mapsto (\exp(2\pi i t_1), \ldots, \exp(2\pi i t_n))$ (which is also the covering map).

REMARK II.2.9. The above considerations also allow us to define a natural map

$$T_eG \to \mathrm{Vect}(G) \tag{II.2.5}$$

mapping $X \in T_eG$ to the left-invariant vector field $\widetilde{X}$ on G such that $\widetilde{X}_e = X$. Under this identification one can show that

$$\mathrm{ad}(X)Y = [\widetilde{X}, \widetilde{Y}]_e. \tag{II.2.6}$$

Thus the Lie bracket on T_eG is the restriction of the Lie bracket of vector fields on G to the subspace of left-invariant vector fields. The skew symmetry and Jacobi identity of $\mathrm{ad}(X)Y$ now follow from the corresponding properties of the Lie bracket on vector fields.

The following is a formal statement of the 'first principle':

THEOREM II.2.10. *If G is connected, then any homomorphism of Lie groups $R : G \to H$ is determined by its derivative $\rho = dR_e$ at the identity.*

PROOF. By the naturality result expressed by the commutative diagram above,

$$R(\exp(X)) = \exp(\rho(X)) \text{ for all } X \in \mathfrak{g} \tag{II.2.7}$$

Since exp is a local diffeomorphism, this means that $R(g)$ is determined by ρ for all g in a small neighbourhood U of e in G. (Some authors write log for a local inverse of exp. Then we'd have $R(g) = \exp(\rho(\log g))$.)

If G' be the group generated by U, then R is also determined by ρ on G'. We claim, however, that $G' = G$. To see this, note that G' is clearly open from its definition. But then all its cosets must also be open and so G', being the complement in G' of the union of all cosets not equal to G, must be closed. So G' is open and closed, and so is the whole of G by connectedness. □

REMARK II.2.11. More generally, if G were not connected, the $\mathrm{d}R_e$ determines R on the identity component of G.

From the definition of the Lie algebra structure on $\mathfrak{g}$ and $\mathfrak{h}$, we've seen that ρ must be a map of Lie algebras. The circle would be completed by showing that every such map arises as the derivative of Lie group homomorphism. This is not quite true: the homomorphisms $S^1 \to S^1$ are all of the form $R : e^{i\theta} \mapsto e^{in\theta}$, where $n \in \mathbb{Z}$, giving rise to linear maps $\rho : t \mapsto nt$. Thus the linear map $t \mapsto at$ appears as the derivative of a homomorphism if and only if a is an integer. The problem turns out to be that S^1 is not simply connected: Fulton and Harris show that if G is simply connected, then every map of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{h}$ arises as the derivative of a homomorphism $G \to H$ of Lie groups.

DEFINITION II.2.12. A Lie subgroup of a Lie group is a closed embedded submanifold $H \subset G$ which is also a subgroup.

Fortunately, the following 'closed subgroup theorem' shows that we if H is a subgroup that is closed in G, then it is automatically a Lie group.

THEOREM II.2.13. *A closed subgroup H of a Lie group G is a submanifold, and hence a Lie subgroup.*

PROOF. We give the main ideas rather than all details. The reader is referred to Adams, Theorem 2.27, p. 17, for the full story.

We do not know that H is smooth anywhere, but we can construct a candidate tangent space as follows. Denoting by log a local inverse of the exponential map of G, let U be a neighbourhood of e in H which is so small that log is defined on U. If H were smooth, then $\log(U)$ would be a neighbourhood of 0 in T_eH. Since we can't assume any smoothness, we extract a candidate tangent space for H at e by defining

$$S = \{X \in T_eG \text{ arising as limits } X = \lim_{n\to\infty} X_n/|X_n|, X_n \in \log(U)\}. \tag{II.2.8}$$

where $|X_n|$ is defined by any euclidean structure on T_eG, and let

$$W = \{tX : t \in \mathbb{R}, X \in S\}. \tag{II.2.9}$$

If H were smooth at e, then S would be the unit sphere in T_eH and W would be T_eH itself.

Having made the definition, one uses the group structure and closedness of H to prove that $\exp(W) \subset H$ and that W is a vector space. For the first of these, suppose that $X \in S$. Then for fixed $t \in \mathbb{R}$, tX is approximated by $tX_n/|X_n|$, with $\exp(X_n) \in H$ and $|X_n| \to 0$. Pick $N_n \to \infty$ so that $N_n|X|_n \to t$ (and all N_n integers). Then

$$N_nX_n = N_n|X_n|\frac{X_n}{|X_n|} \to tX \text{ for } n \to \infty.$$

But $\exp(N_nX_n) = \exp(X_n)^{N_n} \in H$ for all n. By closedness of H, $\exp(tX) \in H$ as required.

For the second part, suppose X and Y are in W and $X + Y \neq 0$. We need to see that $X + Y \in W$. One can show that

$$\exp(tX)\exp(tY) = \exp(t(X + Y) + O(t^2)) \tag{II.2.10}$$

using just the definition of the exponential map. If we set $t = 1/n$ on the RHS, we have

$$Z_n = \frac{X+Y}{n} + O(1/n^2) \text{ so } Z_n/|Z_n| \to X + Y$$

and $\exp(Z_n) \in H$ (since $\exp(tX)$ and $\exp(tY)$ are in H). Thus $X+Y \in S \subset W$. Finally, one shows that $\exp|W$ is a local diffeomorphism onto a neighbourhood in H of e. $\square$

This result gives uniform proofs of the statements (some of them proved) in Chapter I, about the well-known subgroups of GL_n being themselves Lie groups. Thus we have

- $\mathrm{SL}_n(K)$ is a closed subgroup of $\mathrm{GL}_n(K)$, and hence is a Lie group. The Lie algebra consists of the trace free $n \times n$ matrices over K (K is $\mathbb{R}$ or $\mathbb{C}$).
- O_n and SO_n are Lie groups (closed subgroups of $\mathrm{GL}_n(\mathbb{R})$). The Lie algebras consist of the skew-symmetric (respectively trace-free skew-symmetric real $n\times n$ matrices)
- U_n and SU_n are Lie groups (closed subgroups of $\mathrm{GL}_n(\mathbb{C})$). The Lie algebras consist of the skew-adjoint (respectively trace-free skew-adjoint $n \times n$ complex matrices).

Exercise II.2.1. Calculate the real dimensions of O_n, SO_n, U_n, SU_n from their Lie algebras.

2.1. Classification of abelian Lie groups. The results in the previous subsection can be put together to classify abelian Lie groups.

Theorem II.2.14. *A connected abelian Lie group G must be a product of the form $\mathbb{T}^n \times \mathbb{R}^m$.*

Proof. The key point is that if G is abelian, then exp is a group homomorphism. In general, for X and Y sufficiently small,

$$\exp(X)\exp(Y) = \exp(X + Y + O(|X|^2 + |Y|^2)). \tag{II.2.11}$$

Suppose now that G is abelian and take N large. Then for fixed X and Y,

$$\begin{aligned} \exp(X)\exp(Y) &= \exp(X/N)^N \exp(Y/N)^N \\ &= (\exp(X/N)\exp(Y/N))^N \\ &= \exp(X/N + Y/N + O(1/N^2))^N \\ &= \exp(X + Y + O(1/N)). \end{aligned} \tag{II.2.12}$$

Letting N go to ∞, we have $\exp(X)\exp(Y) = \exp(X + Y)$ as required.

From the proof of the 'first principle' above, since exp is now a homomorphism, $\exp(T_eG)$ is a subgroup of G and it is the subgroup generated by a neighbourhood of e. It is therefore the connected component of e in G, which is all of G, as G is connected.

Since exp is a local diffeomorphism, the kernel K of exp is a discrete subgroup and $G = T_eG/K$.

One can show by induction that the only discrete subgroups of $\mathbb{R}^k$ are lattices. Therefore K is a lattice and if the rank is n, it follows that $G = \mathbb{R}^k/K$ is the product of a torus of dimension n and a real vector space of dimension $k - n$. □

Corollary II.2.15. *Every compact connected abelian Lie group is a torus. If G is a compact connected Lie group and H is an abelia subgroup, then it is a torus.*