Example 1.13. (1) Let $H \leq G$ be a subgroup and let G/H be the set of left cosets. Then $\mathbb{Z}[G/H]$ is a permutation module.
(2) Let $\Omega = \sqcup_{i \in I} \Omega_i$ (disjoint union). Then $\mathbb{Z}\Omega = \oplus_{i \in I} \mathbb{Z}\Omega_i$.

In particular, every permutation module can be expressed as

$$\mathbb{Z}\Omega = \bigoplus_{\omega \in \Omega^0} \mathbb{Z}[G/G_\omega],$$

where Ω^0 is a system of representatives of the orbits of the G-action and $G_\omega = \{g \in G \mid g\omega = \omega\}$ is the stabiliser (or isotropy group) of ω. We say G acts **freely** on Ω if all stabilisers are trivial.

Lemma 1.14. *Let Ω be a free G-set and let Ω^0 be a system of representatives for the G-orbits. Then $\mathbb{Z}\Omega$ is a free G-module with basis Ω^0.*

Lemma 1.15. *Let $H \leq G$ be a subgroup of G. Then $\mathbb{Z}G$ is free as a left H-module.*

Now let us define the cohomology groups:

Definition 1.16. Let G be a group. Then the n-th cohomology group of G with coefficients in the G-module M is defined to be

$$\mathrm{H}^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M).$$

In chapter one we have determined the zeroth cohomology group Ext^0. Hence

$$\mathrm{H}^0(G, M) \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \cong M^G,$$

where M^G denote the G-fixed points of M. We have, so far, computed cohomology via projective resolutions and defined the projective resolution of a module M to be the shortest length of a projective resolution of M. One theme of this course will be cohomological finiteness conditions for groups, so let's make first definition.

Definition 1.17. Let G be a group. The cohomological dimension of G, denoted $\mathrm{cd}G$ is defined to be

$$\mathrm{cd}G = \mathrm{pd}_{\mathbb{Z}G}\mathbb{Z}.$$

The above Lemma 1.15 implies directly:

Proposition 1.18. *Let $H \leq G$ be a subgroup of G. Then*

$$\mathrm{cd}H \leq \mathrm{cd}G.$$

Remark 1.19. One can, of course always define the group ring RG for any ring R. $\mathrm{H}^*_R(G, -)$ and $\mathrm{cd}_R G$ are defined analogously. Something more here, adjoint functors?

We shall now spend some time on finding projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$. Let us begin with two easy examples:

Example 1.20. (1) Let $G = \langle x \rangle$ be an infinite cyclic group. Then

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{*(x-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a projective (free) resolution of $\mathbb{Z}$.

(2) Let G be a cyclic group of order n generated by t. Then, as seen before, $\mathbb{Z}G \cong \mathbb{Z}[t]/(T^n - 1)$. Now $T^n - 1 = (T-1)(T^{n-1} + T^{n-2} + ... + T^o)$ and hence for each $x \in \mathbb{Z}G$ it follows that

$$(t-1)x = 0 \iff x = (t^{n-1} + ... + t + t^0)y = Ny \quad \text{some } y \in \mathbb{Z}G.$$

Hence there is a projective (free) resolution of $\mathbb{Z}$ of infinite length:

$$... \xrightarrow{*(t-1)} \mathbb{Z}G \xrightarrow{*N} \mathbb{Z}G \xrightarrow{*(t-1)} ... \xrightarrow{*N} \mathbb{Z}G \xrightarrow{*(t-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

We will see later that G has no projective resolution of finite length.

Corollary 1.21. *Let G be a group with* cd$G < \infty$*, then G is torsion-free.*

3. Finitely generated resolutions

We already did see one cohomological finiteness condition, the cohomological dimension of a group. The main purpose of this chapter is a discussion of the notion of groups of type FP_n, which can be viewed as a generalisation of finite generation (at least as long as $n \geq 2$).

Definition 1.22. Let R be a ring.

(1) Let M be an R-module. We say M is of type FP_n if there is a projective resolution $P_* \twoheadrightarrow M$ with P_i finitely generated for all $i \leq n$.
(2) M is of type FP_∞ if there is a projective resolution $P_* \twoheadrightarrow M$ with P_i finitely generated for all $n \geq 0$.
(3) M is of type FP if M is of type FP_∞ and $\text{pd}_R M < \infty$.

Remark 1.23. (1) M is of type FP_0 if and only if M is finitely generated.
(2) M is of type FP_1 if and only if M is finitely presented.
(3) Let M be of type FP_n . Then there is a free resolution $F_* \twoheadrightarrow M$ with each F_i finitely generated for all $i \leq n$.

We say a module is of type FL if there is a finite length free resolution $F_* \twoheadrightarrow M$ where all F_i are finitely generated. It is obvious that modules of type FL are of type FP but the converse is not necessarily true.

Definition 1.24. A group G is said to be of type FP_n if $\mathbb{Z}$ is a $\mathbb{Z}G$-module of type FP_n .

Remark 1.25. Every group is of type FP_0, since the augmentation map $\epsilon : \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ gives the beginning of a projective resolution and $\mathbb{Z}G$ is a finitely generated $\mathbb{Z}G$-module.

Proposition 1.26. *A group G is of type* FP_1 *if and only if G is finitely generated.*

The description of groups of type FP_2 is already a lot more complicated. A group is called almost finitely presented if there is an exact sequence of groups $K \hookrightarrow F \twoheadrightarrow G$ where F is finitely generated free and $K/[K,K]$ is finitely generated as a G-module. Finitely presented groups are almost finitely presented but the converse is not true in general, see the examples by Bestvina and Brady [**1**]. Bieri [**2**] has shown that the property FP_2 is equivalent to the group being almost finitely presented.

Now let's have a look at finite extensions. We cannot make any more general statements as even finite generation is in general not a subgroup-closed property.

Proposition 1.27. *Let $G' \leq G$ be a subgroup of finite index. Then G is of type* FP_n *if and only if G' is of type* FP_n .

Definition 1.28.
(1) A group G is of type FP iff G is of type FP_∞ and $\mathrm{cd}G < \infty$.
(2) A group is of type FL if G has a finite length finitely generated free resolution.

Obviously does FL imply FP but the converse is not known. Let P be a projective module in the top dimension of a projective resolution of $\mathbb{Z}$. Suppose F is a finitely generated free module such that $P \oplus F$ is free. Then on can construct a finitely generated free resolution

$$F \hookrightarrow P \oplus F \to F_{n-1} \to ... \to F_0 \twoheadrightarrow \mathbb{Z}$$

. We say such a P is **stably free.**

Proposition 1.29. *Let G be a group of type* FP *Suppose that*

$$0 \to P \to F_{n-1} \to ... \twoheadrightarrow \mathbb{Z}$$

is a finitely generated resolution with F_i finitely generated for all $i \leq n-1$. Then G is of type FL *if and only of P is stable free.*

Hence the question whether FL implies FP reduces to the question whether there are projectives that are not stably free. Over general rings the answer can be Yes. There are even examples over group rings $\mathbb{Z}G$ where $G = \mathbb{Z}_{23}$ due to Milnor [**13**, Chapter 3]. These groups, however have infinite cohomological dimension.