Applications of Differential Geometry to Mathematical Physics

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Outline

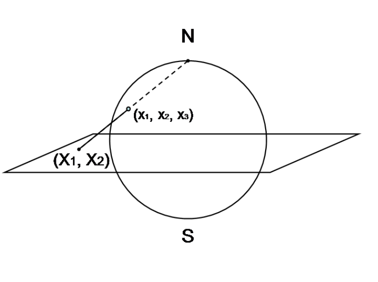
- Manifolds
- Fibre bundles
- Vector bundles, principal bundles
- Sections in fibre bundles
- Metric, Connection
- General Relativity, Yang-Mills theory

The 2-sphere S^2

- $S^2 = \left\{ (x_1, x_2, x_3) : \sum_{i=1}^{3} x_i^2 = 1 \right\}$.
- Polar coordinates:

$$\begin{aligned} x_1 &= \cos\phi \sin\theta, \\ x_2 &= \sin\phi \sin\theta, \\ x_3 &= \cos\theta. \end{aligned}$$

- Problem: We can't label S^2 with single coord system such that
  1. Nearby points have nearby coords.
  2. Every point has unique coords.



- Stereographic Projection

$$\begin{aligned} X_1 &= \frac{x_1}{1 - x_3}, \\ X_2 &= \frac{x_2}{1 - x_3}. \end{aligned}$$

Manifold

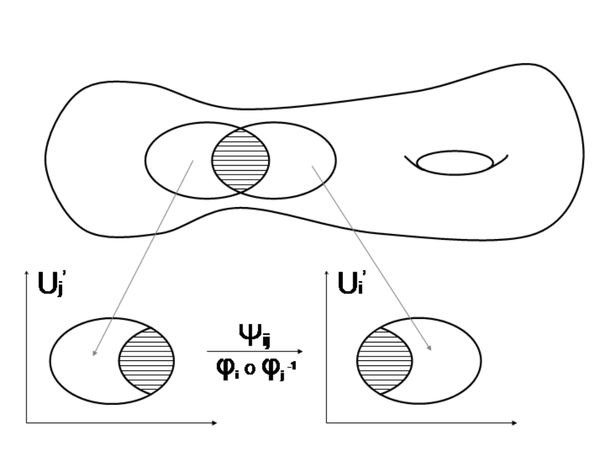
Def: M is an m-dimensional (differentiable) manifold if

- M is a topological space.
- M comes with family of charts $\{(U_i, \phi_i)\}$ known as *atlas.*
- $\{U_i\}$ is family of open sets covering M: $\bigcup_i U_i = M$.
- ϕ_i is homeomorphism from U_i onto open subset U_i' of $\mathbb{R}^m$.
- Given $U_i \cap U_j \neq \emptyset$, then the map

$$\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is C^∞. ψ_{ij} are called *crossover maps.*

Picture



$$\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

Example: S^2

- Projection from North pole:

$$X_1 = \frac{x_1}{1 - x_3}, \qquad X_2 = \frac{x_2}{1 - x_3}.$$

- $U_1 = S^2 \setminus N,\ U_1' = \mathbb{R}^2$:

$$\phi_1 : U_1 \to \mathbb{R}^2 : \quad (x_1, x_2, x_3) \mapsto (X_1, X_2)$$

- Projection from South pole:

$$Y_1 = \frac{x_1}{1 + x_3}, \qquad Y_2 = \frac{x_2}{1 + x_3}.$$

- $U_2 = S^2 \setminus S,\ U_2' = \mathbb{R}^2$:

$$\phi_2 : U_2 \to \mathbb{R}^2 : \quad (x_1, x_2, x_3) \mapsto (Y_1, Y_2)$$

Crossover map $\psi_{21} = \phi_2 \circ \phi_1^{-1}$:

$$\psi_{21} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 : (X_1, X_2) \mapsto (Y_1, Y_2) = \left(\frac{X_1}{X_1^2 + X_2^2}, \frac{-X_2}{X_1^2 + X_2^2} \right).$$

Example: S^3

- Stereographic coords for S^3 work the same way as for S^2, e.g.

$$X_i = \frac{x_i}{1 - x_4},$$

where $i = 1, 2, 3$ for the projection from the "North pole".

- Note S^3 can be identified with $SU(2)$, i.e. complex 2×2 matrices which satisfy

$$U\ U^\dagger = U^\dagger U = 1 \quad \text{and} \quad \det U = 1. \tag{1}$$

Setting

$$U = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

satisfies all the conditions in (1) provided

$$|z_1|^2 + |z_2|^2 = 1,$$

which is the equation for S^3.

The trivial bundle

- Often manifolds can be build up from smaller manifolds.
- An important example is the Cartesian product: $E = B \times F$ (known as *trivial bundle*)
- *Fibre bundles* are manifolds which look like Cartesian products, *locally*, but not *globally*.
- This concept is very useful for physics. Non-trivial fibre bundles occur for example in general relativity, but also due to boundary conditions "at infinity".

Fibre bundle

Def: A *fibre bundle* (E, π, M, F, G) consists of

- A manifold E called *total space*, a manifold M called *base space* and a manifold F called *fibre* (or typical fibre)
- A surjection $\pi : E \to M$ called the *projection.* The inverse image of a point $p \in M$ is called the fibre at p, namely $\pi^{-1}(p) = F_p \cong F$.
- A Lie group G called *structure group* which acts on F on the left.
- A set of open coverings $\{U_i\}$ of M with diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$, such that $\pi \circ \phi_i(p, f) = p$. The map is called the *local trivialization,* since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ to $U_i \times F$.
- Transition functions $t_{ij} : U_i \cap U_j \to G$, such that $\phi_j(p, f) = \phi_i(p, t_{ij}(p)f)$. Fix p then $t_{ij} = \phi_i^{-1} \circ \phi_j$.

The importance of the transition functions

- Consistency conditions (ensure $t_{ij} \in G$)

$$\begin{array}{ll} t_{ii}(p) = e & p \in U_i \\ t_{ij}(p) = t_{ji}^{-1}(p) & p \in U_i \cap U_j \\ t_{ij}(p) \cdot t_{jk}(p) = t_{ik}(p) & p \in U_i \cap U_j \cap U_k \end{array}$$

- If all transition functions are the identity map e, then the fibre bundle is called the trivial bundle, $E = M \times F$.
- The transition functions of two local trivializations $\{\phi_i\}$ and $\{\tilde{\phi}_i\}$ for fixed $\{U_i\}$ are related via

$$\tilde{t}_{ij}(p) = g_i^{-1}(p) \cdot t_{ij}(p) \cdot g_j(p).$$

  where for fixed p, we define $g_i : F \to F : g_i = \phi_i^{-1} \circ \tilde{\phi}_i$.
- For the trivial bundle, $t_{ij}(p) = g_i^{-1}(p) \cdot g_j(p)$.

Tangent vectors

- Given a curve $c : (-\epsilon, \epsilon) \to M$ and a function $f : M \to \mathbb{R}$, we define the tangent vector $X[f]$ at $c(0)$ as directional derivative of $f(c(t))$ along $c(t)$ at $t = 0$, namely

$$X[f] = \left. \frac{df(c(t))}{dt} \right|_{t=0}.$$

- In local coords, this becomes

$$\frac{\partial f}{\partial x^\mu} \left. \frac{dx^\mu(c(t))}{dt} \right|_{t=0},$$

hence

$$X[f] = X^\mu \left(\frac{\partial f}{\partial x^\mu} \right).$$

- To be more mathematical, the tangent vectors are defined via equivalence classes of curves.

More about Tangent vectors

- Vectors are independent of the choice of coordinates, hence

$$X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial y^\mu}.$$

- The components of X^μ and $\tilde{X}^\mu$ are related via

$$\tilde{X}^\mu = X^\nu \frac{\partial y^\mu}{\partial x^\nu}.$$

- It is very useful to define the pairing

$$\left\langle dx^\nu, \frac{\partial}{\partial x^\mu} \right\rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu.$$

- This leads us to one-forms $\omega = \omega_\mu dx^\mu$, also independent of choice of coordinates. Now, we have

$$\omega = \omega_\mu dx^\mu = \tilde{\omega}_\nu dy^\nu \quad \Longrightarrow \quad \tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial y^\nu}.$$

- This can be generalized further to tensors $T^{\mu_1 \cdots \mu_q}{}_{\nu_1 \ldots \nu_r}$.

The Tangent bundle

- At each point $p \in M$ all the tangent vectors form an n dimensional vector space T_pM, so tangent vectors can be added and multiplied by real numbers:

$$\alpha X_1 + \beta X_2 \in T_pM \quad \text{for } \alpha, \beta \in \mathbb{R}, \quad X_1, X_2 \in T_pM.$$

- A basis of T_pM is given by $\partial/\partial x^\mu$, $(1 \leq \mu \leq n)$, hence $\dim M = \dim T_pM$.
- The union of all tangent spaces forms the tangent bundle

$$TM = \bigcup_{p \in M} T_pM.$$

- TM is a $2n$ dimensional manifold with base space M and fibre $\mathbb{R}^n$. It is an example of a *vector bundle.*

The Tangent bundle TS^2

- We use the two stereographic projections as our charts.
- The coords $(X_1, X_2) \in U_1'$ and $(Y_1, Y_2) \in U_2'$ are related via

$$Y_1 = \frac{X_1}{X_1^2 + X_2^2}, \quad Y_2 = \frac{-X_2}{X_1^2 + X_2^2}.$$

- Given $u \in TS^2$ with $\pi(u) = p \in U_1 \cap U_2$, then the local trivializations ϕ_1 and ϕ_2 satisfy $\phi_1^{-1}(u) = (p, V_1^\mu)$ and $\phi_2^{-1}(u) = (p, V_2^\mu)$. The local trivialization is

$$t_{12} = \frac{\partial(Y_1, Y_2)}{\partial(X_1, X_2)} = \frac{1}{(X_1^2 + X_2^2)^2} \begin{pmatrix} X_2^2 - X_1^2 & -2X_1X_2 \\ -2X_1X_2 & X_1^2 - X_2^2 \end{pmatrix}.$$

- Check: $t_{21}(p) = t_{12}^{-1}(p)$.

Example: $U(1)$ bundle over S^2

- Consider a fibre bundle with fibre $U(1)$ and base space S^2.
- Let $\{U_N, U_S\}$ be an open covering of S^2 where

$$\begin{aligned} U_N &= \{(\theta,\phi) : 0 \leq \theta < \pi/2 + \epsilon, 0 \leq \phi < 2\pi\} \\ U_S &= \{(\theta,\phi) : \pi/2 - \epsilon < \theta \leq \pi, 0 \leq \phi < 2\pi\} \end{aligned}$$

- The intersection $U_N \cap U_S$ is a strip which is basically the equator. Local trivializations are

$$\phi_N^{-1}(u) = (p, e^{i\alpha_N}), \quad \phi_S^{-1}(u) = (p, e^{i\alpha_S})$$

  where $p = \pi(u)$.
- Possible transition functions are $t_{NS} = e^{in\phi}$, where $n \in \mathbb{Z}$.
- The fibre coords in $U_N \cap U_S$ are related via

$$e^{i\alpha_N} = e^{in\phi} e^{i\alpha_S}.$$

- If $n = 0$ this is the trivial bundle $P_0 = S^2 \times S^1$. For $n \neq 0$ the $U(1)$ bundle P_n is twisted.

Magnetic monopoles and the Hopf bundle

- P_n is an example of a *principal bundle* because the fibre is the same as the structure group $G = U(1)$.
- In physics, P_n is interpreted as a magnetic monopole of charge n.
- Given $S^3 = \left\{x \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\}$ we can define the Hopf map: $\pi : S^3 \to S^2$ by

$$\begin{aligned} \xi_1 &= 2(x_1 x_3 + x_2 x_4) \\ \xi_2 &= 2(x_2 x_3 - x_1 x_4) \\ \xi_3 &= x_1^2 + x_2^2 - x_3^2 - x_4^2. \end{aligned}$$

  which implies $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$.
- It turns out that with this choice of coords S^3 can be identified with P_1, a nontrivial $U(1)$ bundle over S^2, known as the *Hopf bundle*.

Sections

Def: Let (E, M, π) be a fibre bundle. A section $s : M \to E$ is a smooth map which satisfies $\pi \circ s = id_M$. Here, $s|_p$ is an element of the fibre $F_p = \pi^{-1}(p)$. The space of section is denoted by $\Gamma(E)$.

- A local section is defined on $U \subset M$, only.
- Note that not all fibre bundles admit global sections!
- Example: The wave function $\psi(\mathbf{x}, t)$ in quantum mechanics can be thought of as a section of a complex line bundle $E = \mathbb{R}^{3,1} \times \mathbb{C}$.
- Vector fields associate a tangent vector to each point in M. They can be thought of as sections of TM.

More on sections

- Vector bundles always have at least one section, the null section s_0 with

$$\phi_i^{-1}(s_0(p)) = (p, 0)$$

in any local trivialization.

- A principal bundle E only admits a global section if it is trivial: $E = M \times F$.
- A section in a principal bundle can be used to construct the trivialization of the bundle which uses that we can define a right action which is independent of the local trivialization:

$$ua = \phi(p, g_i a), \quad a \in G$$

Associated Vector bundle

- Given a principal fibre bundle $P(M, G, \pi)$ and a k-dimensional vector space V, and let ρ be a k dimensional representation of G then the *associated vector bundle* $E = P \times_\rho V$ is defined by identifying the points

$$(u, v) \quad \text{and} \quad (ug, \rho(g)^{-1}v) \in P \times V$$

  where $u \in P$, $g \in G$, and $v \in V$.
- The projection $\pi_E : E \to M$ is defined by $\pi_E(u, v) = \pi(u)$, which is well defined because

$$\pi_E(ug, \rho(g)^{-1}v) = \pi(ug) = \pi(u) = \pi_E(u, v)$$

- The transition functions of E are given by $\rho(t_{ij}(p))$ where $t_{ij}(p)$ are the transition functions of P.
- Conversely, a vector bundle naturally induces a principal bundle associated with it.

Metric

- Manifolds can carry further structure, for example a *metric.*
- A metric g is a $(0,2)$ tensor which satisfies at each point $p \in M$:
  1. $g_p(U,V) = g_p(V,U)$
  2. $g_p(U,U) \geq 0$, with equality only when $U = 0$.

  where $U, V \in T_pM$.
- The metric g provides an inner product for each tangent space T_pM.
- Notation:
$$g = g_{\mu\nu} dx^\mu dx^\nu .$$
- If M is a submanifold of N with metric g_N and $f : M \to N$ is the embedding map, then the induced metric g_M is
$$g_{M\mu\nu}(x) = g_{N\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} .$$

Connection for the Tangent bundle

- Consider the "derivative" of a vector field $V = V^\mu \frac{\partial}{\partial x^\mu}$ w.r.t. x^ν :

$$\frac{\partial V^\mu}{\partial x^\nu} = \lim_{\triangle x \to 0} \frac{V^\mu(\ldots, x^\nu + \triangle x^\nu, \ldots) - V^\mu(\ldots, x^\nu, \ldots)}{\triangle x^\nu}$$

- This doesn't work as the first vector is defined at $x + \triangle x$ and the second at x.
- We need to transport the vector V^μ from x to $x + \triangle x$ "without change". This is known as *parallel transport*.
- This is achieved by specifying a *connection* $\Gamma^\mu{}_{\nu\lambda}$, namely the parallel transported vector $\tilde{V}^\mu$ is given by

$$\tilde{V}^\mu(x + \triangle x) = V^\mu(x) - V^\lambda(x)\Gamma^\mu{}_{\nu\lambda}(x)\triangle x^\nu.$$

- The covariant derivative of V w.r.t. x^ν is

$$\lim_{\triangle x^\nu \to 0} \frac{V^\mu(x + \triangle x) - \tilde{V}^\mu(x + \triangle x)}{\triangle x^\nu} = \frac{\partial V^\mu}{\partial x^\nu} + V^\lambda \Gamma^\mu{}_{\nu\lambda}.$$

The Levi-Civita Connection

- We demand that the metric g is covariantly constant.
- This means, if two vectors X and Y are parallel transported along any curve, then the inner product $g(X, Y)$ remains constant.
- The condition
$$\nabla_V(g(X,Y)) = 0,$$
gives us the Levi-Civita connection.
- The Levi-Civita connection can be written as
$$\Gamma^{\kappa}{}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right).$$

General Relativity

- The Levi-Civita connection doesn't transform like a tensor. However, from it, we can build the curvature tensor:

$$R^{\kappa}{}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\kappa}{}_{\nu\lambda} - \partial_{\nu}\Gamma^{\kappa}{}_{\mu\lambda} + \Gamma^{\eta}{}_{\nu\lambda}\Gamma^{\kappa}{}_{\mu\eta} - \Gamma^{\eta}{}_{\mu\lambda}\Gamma^{\kappa}{}_{\nu\eta}.$$

- Important contractions of the curvature tensor are the *Ricci tensor* *Ric* :

$$Ric_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}.$$

  and the *scalar curvature* $\mathcal{R}$:

$$\mathcal{R} = g^{\mu\nu} Ric_{\mu\nu}.$$

- Now, we have the ingredients for *Einstein's Equations of General Relativity*, namely

$$Ric_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu},$$

  where G is the gravitational constant and $T_{\mu\nu}$ is the energy momentum tensor which describes the distribution of matter.

Yang-Mills theory

- An example of Yang-Mills theory is given by the following Lagrangian density,

$$\mathcal{L} = \frac{1}{8}\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right) + \frac{1}{2}\left(D_\mu\Phi\right)^\dagger D^\mu\Phi - U(\Phi^\dagger\Phi). \qquad (2)$$

where

$$D_\mu\Phi = \partial_\mu\Phi + \mathbf{A}_\mu\Phi \quad \text{and} \quad \mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + \left[\mathbf{A}_\mu, \mathbf{A}_\nu\right].$$

- Here Φ is a two component complex scalar field.
- $\mathbf{A}_\mu$ is called a gauge field and is $su(2)$-valued, i.e. $\mathbf{A}_\mu$ are anti-hermitian 2×2 matrices.
- $\mathbf{F}_{\mu\nu}$ is known as the field strength (also $su(2)$-valued)
- This Lagrangian is Lorentz invariant.

Gauge invariance

Lagrangian (2) is also invariant under local gauge transformations:

- Let $g \in SU(2)$ be a space-time dependent gauge transformation with

$$\Phi \mapsto g\Phi, \quad \text{and} \quad \mathbf{A}_\mu \mapsto g\mathbf{A}_\mu g^{-1} - \partial_\mu g g^{-1}.$$

- The covariant derivative $D_\mu\Phi$ transforms as

$$\begin{aligned} D_\mu\Phi &\mapsto \partial_\mu(g\Phi) + \left(g\mathbf{A}_\mu g^{-1} - \partial_\mu g g^{-1}\right) g\Phi \\ &= gD_\mu\Phi \end{aligned}$$

- Hence $\Phi^\dagger\Phi \mapsto (g\Phi)^\dagger g\Phi = \Phi^\dagger g^\dagger g\Phi = \Phi^\dagger\Phi$, and similarly for $(D_\mu\Phi)^\dagger D^\mu\Phi$.
- Finally, $\mathbf{F}_{\mu\nu} \mapsto g\mathbf{F}_{\mu\nu}g^{-1}$, so

$$\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right)$$

is also invariant

Yang-Mills theory and Fibre bundles

In a more mathematical language:

- The gauge field $\mathbf{A}_\mu$ corresponds to the connection of the principal $SU(2)$ bundle.
- The field strength $\mathbf{F}_{\mu\nu}$ corresponds to the curvature of the principal $SU(2)$ bundle.
- The complex scalar field Φ is a section of the associated $\mathbb{C}^2$ vector bundle.
- The action of $g \in SU(2)$ on Φ and $\mathbf{A}_\mu$ is precisely what we expect for an associated fibre bundle.
- Surprisingly, mathematicians and physicist derived the same result very much independently!