More Applications of Differential Geometry to Mathematical Physics

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Outline

- Review: Manifolds, Fibre bundles
- Differential forms and integration
- The Hodge $*$ and products of p-forms
- Complex Geometry

Manifold

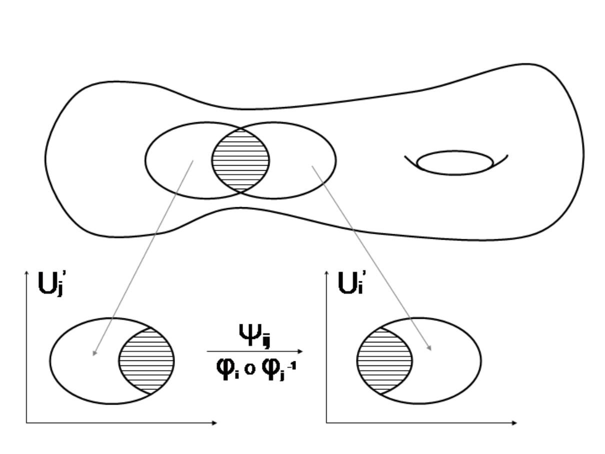
Def: M is an m-dimensional (differentiable) manifold if

- M is a topological space.
- M comes with family of charts $\{(U_i, \phi_i)\}$ known as *atlas.*
- $\{U_i\}$ is family of open sets covering M: $\bigcup_i U_i = M$.
- ϕ_i is homeomorphism from U_i onto open subset U_i' of $\mathbb{R}^m$.
- Given $U_i \cap U_j \neq \emptyset$, then the map

$$\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

  is C^∞. ψ_{ij} are called *crossover maps*.

Picture



$$\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

Functions between manifolds

- Let M be an m dimensional manifold with charts $\phi_i : U_i \to \mathbb{R}^m$ and N be an n dimensional manifold with charts $\psi_j : \tilde{U}_j \to \mathbb{R}^n$.
- Let f be a map between manifolds:

$$f : M \to N, p \mapsto f(p).$$

- This has a coordinate presentation

$$F_{ji} = \psi_j \circ f \circ {\phi_i}^{-1} : \mathbb{R}^m \to \mathbb{R}^n, x \mapsto \psi_j(f(\phi_i^{-1}(x))),$$

  where $x = \phi_i(p)$ ($p \in U_i$ and $f(p) \in \tilde{U}_j$).
- Using the coordinate presentation all the calculus rules in $\mathbb{R}^n$ work for maps between manifolds. If the presentations F_{ji} are differentiable in all charts then f is differentiable.

Fibre bundle

Def: A *fibre bundle* (E, π, M, F, G) consists of

- A manifold E called *total space*, a manifold M called *base space* and a manifold F called *fibre* (or typical fibre)
- A surjection $\pi : E \to M$ called the *projection.* The inverse image of a point $p \in M$ is called the fibre at p, namely $\pi^{-1}(p) = F_p \cong F$.
- A Lie group G called *structure group* which acts on F on the left.
- A set of open coverings $\{U_i\}$ of M with diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$, such that $\pi \circ \phi_i(p, f) = p$. The map is called the *local trivialization,* since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ to $U_i \times F$.
- Transition functions $t_{ij} : U_i \cap U_j \to G$, such that $\phi_j(p, f) = \phi_i(p, t_{ij}(p)f)$. Fix p then $t_{ij} = \phi_i^{-1} \circ \phi_j$.

Recall Tangent vectors

- Tangent vectors act on functions via

$$X[f] = X^\mu \frac{\partial f}{\partial x^\mu} \quad \text{(sum over repeated indices)}$$

- The components of X^μ and $\tilde{X}^\mu$ are related via

$$\tilde{X}^\mu = X^\nu \frac{\partial y^\mu}{\partial x^\nu} \quad \text{(Einstein's summation convention again)}$$

- We defined the pairing

$$\left\langle dx^\nu, \frac{\partial}{\partial x^\mu} \right\rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu.$$

- This leads us to one-forms $\omega = \omega_\mu dx^\mu$, also independent of choice of coordinates. Now, we have

$$\omega = \omega_\mu dx^\mu = \tilde{\omega}_\nu dy^\nu \quad \Longrightarrow \quad \tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial y^\nu}.$$

Tangent bundle and Cotangent bundle

- A basis of T_pM is given by $\partial/\partial x^\mu$, $(1 \leq \mu \leq n)$, hence $\dim M = \dim T_pM$, and similarly for T_p^*M with basis dx^μ.
- The union of all tangent spaces forms the tangent bundle

$$TM = \bigcup_{p\in M} T_pM.$$

- Similarly, the union of all cotangent spaces forms the cotangent bundle

$$T^*M = \bigcup_{p\in M} T_p^*M.$$

- TM and T^*M are $2n$ dimensional manifolds with base space M and fibre $\mathbb{R}^n$.

Pushforward and Pullback

- Given a smooth map between manifolds

$$f : M \to N, p \mapsto f(p)$$

  we can define a map between the tangent spaces TM and TN via

$$f_* : T_pM \to T_{f(p)}N, V \mapsto f_*V$$

  which is called **pushforward**. Let $g \in C^\infty(N)$ then $g \circ f \in C^\infty(M)$. Define the action of the vector f_*V on g via

$$f_*V(g) = V(g \circ f).$$

- Similarly, we can define a map between the cotangent spaces T^*N and T^*M via

$$f^* : {T^*}_{f(p)}N \to {T^*}_pM, \omega \mapsto f^*\omega$$

  which is called **pullback**. The pullback can be defined via the pairing

$$\langle f^*\omega, V\rangle_M = \langle \omega, f_*V\rangle_N.$$

Metric

- A metric g is a $(0, 2)$ tensor which satisfies at each point $p \in M$:
  1. $g_p(U, V) = g_p(V, U)$ (symmetric)
  2. $g_p(U, U) \geq 0$, with equality only when $U = 0$ (non-degenerate)

  where $U, V \in T_pM$.
- The metric g provides an inner product for each tangent space T_pM.
- Notation:

$$g = g_{\mu\nu} dx^{\mu} dx^{\nu}.$$

- The metric provides an isomorphism between vector fields $X \in TM$ and 1-forms $\eta \in T^*M$ via

$$g(., X) = \eta_X$$

- In physics notation $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ lower and raise indices.

Symplectic form

- A symplectic form ω is a 2-form which satisfies
  1. ω is closed, i.e. $d\omega = 0$.
  2. ω is non-degenarate: $\omega(U, V) = 0$ for all V implies $U = 0$.

  where $U, V \in T_pM$.
- Notation:
$$\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu .$$
- The symplectic form also provides an isomorphism between vector fields $X \in TM$ and 1-forms $\eta \in T^*M$ via
$$\omega(., X) = \eta_X$$

Differential forms

- A basis for a p-form $\in \Omega^p(M)$ is

$$\langle dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \rangle \quad \text{where} \quad 1 \leq \mu_1 < \cdots < \mu_k \leq n.$$

- *Wedge product:*

$$\wedge : \Omega^k \times \Omega^l \to \Omega^{k+l},$$

  where

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha.$$

- *Exterior derivative:* Given

$$\omega = \frac{1}{k!} \omega_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$$

  then

$$d\omega = \frac{1}{k!} \left(\frac{\partial}{\partial \nu} \omega_{\mu_1 \ldots \mu_k} \right) dx^{\nu} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}.$$

- Recall $d^2 = 0$.

Integration on Manifolds

- Recall under change of basis one-forms transform as

$$\tilde{\omega}_\nu = \omega_\mu \left(\frac{\partial x^\mu}{\partial y^\nu} \right)$$

Two chart define the same orientation provided that

$$\det \left(\frac{\partial x^\mu}{\partial y^\nu} \right) > 0.$$

- A manifold is orientable if for any overlapping charts U_i and U_j there exist local coordinates $x^\mu \in U_i$ and $y^\mu \in U_j$ such that $\det \left(\frac{\partial x^\mu}{\partial y^\nu} \right) > 0$.
- The invariant volume element on M is given by

$$\Omega = \sqrt{|g|} dx^1 \wedge \ldots dx^m \quad \text{where} \quad g = \det(g_{\mu\nu}).$$

Integration on Manifolds II

- Now, we can integrate a function $f : M \to \mathbb{R}$ over M. First consider one chart:

$$\int_{U_i} f\Omega = \int_{\phi(U_i)} f(\phi_i^{-1}(x))\sqrt{|g(\phi^{-1}(x)|}dx^1 dx^2 \dots dx^m.$$

- A partition of unity is a family of differentiable functions $\epsilon_i(p)$, $1 \leq i \leq k$ such that
  1. $0 \leq \epsilon_i(p) \leq 1$.
  2. $\epsilon_i(p) = 0$ if $p \notin U_i$
  3. $\epsilon_1(p) + \cdots + \epsilon_k(p) = 1$ for any point $p \in M$.
- Integrate over the whole manifold M via

$$\int_M f\Omega = \sum_{i=1}^{k} \int_{U_i} f(p)\epsilon_i(p)\Omega.$$

Stokes Theorem

- Let w be a p-form and R a $p+1$ dimensional region in M with boundary ∂R, then
$$\int_R d\omega = \int_{\partial R} \omega.$$
- Special case: $\omega = p\ dx + q\ dy$ in $\mathbb{R}^2$, then
$$d\omega = (\partial_y q - \partial_y p) dx \wedge dy.$$
- Hence,
$$\oint_{\mathcal{C}} (p\ dx + q\ dy) = \iint_R (\partial_y q - \partial_y p) dx dy,$$
which is Green's theorem in the plane.

Examples: Stokes and Divergence Theorem

- In $\mathbb{R}^3$ we have $\omega = f_1 dx + f_2 dy + f_3 dz$, and

  $d\omega = (\partial_y f_3 - \partial_z f_2) dy \wedge dz + (\partial_z f_1 - \partial_x f_3) dz \wedge dx + (\partial_x f_2 - \partial_y f_1) dx \wedge dy,$

  which gives rise to the usual Stokes theorem

  $$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_S (\nabla \wedge \mathbf{f}) \cdot \mathbf{n} \, dS.$$

- If $\omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy$ then

  $$d\omega = (\partial_x f_1 + \partial_y f_2 + \partial_z f_3) \, dx \wedge dy \wedge dz,$$

  which gives rise to the divergence theorem:

  $$\iiint_V \nabla \cdot \mathbf{f} \, dxdydz = \iint_S \mathbf{f} \cdot \mathbf{n} \, dS.$$

Hodge $*$

- Define the totally anti-symmetric tensor

$$\epsilon_{\mu_1\mu_2\ldots\mu_m} = \begin{cases} +1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an even permutation of } (12\ldots m) \\ -1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an odd permutation of } (12\ldots m) \\ 0 & \text{otherwise.} \end{cases}$$

- The Hodge $*$ is a linear map $* : \Omega^r(M) \to \Omega^{m-r}(M)$ which acts on a basis vector of $\Omega^r(M)$ via

$$*(dx^{\mu_1} \wedge \ldots dx^{\mu_r}) = \frac{\sqrt{|g|}}{m!} \epsilon^{\mu_1\ldots\mu_r}{}_{\nu_{r+1}\ldots\nu_m} dx^{\nu_{r+1}} \wedge \cdots \wedge dx^{\nu_m}.$$

- The invariant volume element is

$$*1 = \sqrt{|g|} dx^1 \wedge \ldots dx^m.$$

- Examples for $\mathbb{R}^3$:

$$*1 = dx \wedge dy \wedge dz, *dx = dy \wedge dz, *dy = dz \wedge dx, *dz = dx \wedge dy,$$

$$*dy \wedge dz = dx, *dz \wedge dx = dy, *dx \wedge dy = dz, *dx \wedge dy \wedge dz = 1.$$

Inner product on r-forms

- Assume (M, g) is Riemannian, dim $M = m$ and ω is an r-form, then

$$**\omega = (-1)^{r(m-r)}\omega.$$

- Let

$$\omega = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r} dx^{mu_1} \wedge \cdots \wedge dx^{\mu_r} \text{ and } \eta = \frac{1}{r!}\eta_{\mu_1\ldots\mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r},$$

  then

$$\omega \wedge *\eta = \cdots = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\eta^{\mu_1\ldots\mu_r}\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^m,$$

- We can define an inner product on r-forms via

$$(\omega, \eta) = \int_M \omega \wedge *\eta.$$

- Note: $(\omega, \eta) = (\eta, \omega)$ and this inner product is positive definite $((\alpha, \alpha) \geq 0$ with equality only for $\alpha = 0)$.

Ginzburg-Landau potential

- Ginzburg-Landau vortices on $\mathbb{R}^2$ are minimals of the potential energy

$$V(\phi, A) = \frac{1}{2}\int_{\mathbb{R}^2}\left(\mathrm{d}A \wedge *\mathrm{d}A + \overline{\mathrm{d}_A\phi} \wedge *\mathrm{d}_A\phi + \frac{\lambda}{4}(1-\bar{\phi}\phi)^2 * 1\right),$$

where $\phi : \mathbb{R}^2 \to \mathbb{C}$ is a complex scalar field, $A \in \Omega^1(\mathbb{R}^2)$ is the gauge potential one-form, $\mathrm{d}_A\phi = \mathrm{d}\phi - iA\phi$, and $*$ is the Hodge isomorphism.

- In usual physics notation

$$V = \frac{1}{2}\int\left(\frac{1}{2}F^{ij}F_{ij} + \overline{D^i\phi}D_i\phi + \frac{\lambda}{4}(1-\bar{\phi}\phi)^2\right)\mathrm{d}x^2,$$

where $D_i\phi = \partial_i\phi - ia_i\phi$ and $f_{12} = \partial_1 a_2 - \partial_2 a_1$.

Laplacian on p-forms

- Given the exterior derivative $d : \Omega^{r-1}(M) \to \Omega^r(M)$ we can define the adjoint exterior derivative $d^\dagger : \Omega^r(M) \to \Omega^{r-1}(M)$ via

$$d^\dagger = (-1)^{mr+m+1} * d*$$

- Let (M, g) be compact, orientable and without boundary, and $\alpha \in \Omega^r(M)$, $\beta \in \Omega^{r-1}(M)$ then

$$(d\beta, \alpha) = (\beta, d^\dagger \alpha).$$

- The Laplacian $\triangle : \Omega^r(M) \to \Omega^r(M)$ is define by

$$\triangle = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d.$$

- Example: Laplacian on functions:

$$\triangle f = \cdots = -\frac{1}{\sqrt{|g|}}\, \partial_\nu \left(\sqrt{|g|} g^{\mu\nu} \partial_\mu f \right).$$

Hodge decomposition theorem

- An r-form ω_r is called harmonic if $\triangle w_r = 0$.
- Hodge decomposition theorem:

$$\Omega^r(M) = d\Omega^{r-1}(M) \oplus d^\dagger \Omega^{r+1} \oplus \mathrm{Harm}^r(M)$$

  that is

$$w_r = d\alpha_{r-1} + d^\dagger \beta_{r+1} + \gamma_r$$

  with $\triangle \gamma_r = 0$.
- Note $\mathrm{Harm}^r(M)$ is isomorphic to the de Rham cohomology group $H^r(M)$.

Physics equation in differential geometry notation

- The four Maxwell equations can be written as

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \wedge \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}.$$

  and

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0.$$

  where

$$\mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \wedge \mathbf{A}.$$

- In differential geometry notation we have $F = dA$. The Maxwell equations are

$$dF = 0 \quad \text{and} \quad d^\dagger F = j.$$

Complex Manifolds

- A complex manifold is a manifold such that the crossover maps ψ_{ij} are all holomorphic.
- Recall: Let $z = x + iy$ and $f = u + iv$ then $f(x, y)$ is holomorphic in z provided the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- Examples of complex manifolds are $\mathbb{C}^n$, S^2, T^2, $\mathbb{C}P^n$, $S^{2n+1} \times S^{2m+1}$.

Almost complex structure

- An almost complex structure is a $(1,1)$ tensor which acts on real coordinates as

$$J_p \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial y^\mu}, \quad J_p \frac{\partial}{\partial y^\mu} = -\frac{\partial}{\partial x^\mu}.$$

with ${J_p}^2 = -id_{T_pM}$.

- On complex coordinates vector we have

$$J_p \frac{\partial}{\partial z^\mu} = i \frac{\partial}{\partial z^\mu}, \quad J_p \frac{\partial}{\partial \bar{z}^\mu} = -i \frac{\partial}{\partial \bar{z}^\mu}.$$

(multiplication by i).

Hermitian metrics

- A Hermitian metric is a Riemannian metric which satisfies

$$g_p(J_pX, J_pY) = g_p(X, Y),$$

  i.e. g is compatible with J_p.
- The vector J_pX is orthogonal to X wrt g :

$$g_p(J_pX, X) = g_p(J_p^2X, J_pX) = -g_p(J_pX, X) = 0.$$

- For a Hermitian metric $g_{\mu\nu} = 0$ and $g_{\bar{\mu}\bar{\nu}} = 0$, e.g.

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = g\left(J_p\frac{\partial}{\partial z^\mu}, J_p\frac{\partial}{\partial z^\nu}\right) = g(i\frac{\partial}{\partial z^\mu}, i\frac{\partial}{\partial z^\nu}) = -g_{\mu\nu}.$$

The Kähler form

- Define the tensor field Ω via

$$\Omega_p(X, Y) = g_p(J_p X, Y), \quad X, Y \in T_p M.$$

- Ω is antisymmetric and invariant under J_p :

$$\Omega(X, Y) = -\Omega(Y, X), \quad \Omega(J_p X, J_p Y) = \Omega(X, Y).$$

- Ω is a real form and can be written as

$$\Omega = -i g_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu}.$$

- $\Omega \wedge \cdots \wedge \Omega$ ($\dim_{\mathbb{C}} M$-times) provides a volume form for M.
- If $d\Omega = 0$ then g is a Kähler metric.

Remarks

- For Kähler manifold, the metric g is related to the anti-symmetric Kähler form Ω which can be interpreted as a symplectic 2-form
- Topological solitons of Bogomolny type usually have a "moduli space" of static solutions which is a smooth manifold with a natural Kähler metric (given by the kinetic energy)