

Differential Geometry and Soliton Dynamics

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- Homotopy theory
- Ginzburg-Landau vortices
- Vortices at critical coupling and the vortex moduli space
- Relativistic vortex dynamics
- First order vortex dynamics
- Vortices in different geometries

The Fundamental group $\pi_1(M)$

- Given a manifold M and an interval $I = [0, 1]$ we can define *paths*

$$\alpha : I \rightarrow M : t \mapsto \alpha(t), \text{ where } \alpha(0) = p_0, \alpha(1) = p_1.$$

- A *loop* is a path with $p_0 = p_1$.
- Paths can be multiplied via

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2}, \\ \beta(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

- The constant path is $c(s) = p_0$ for all $s \in I$.
- The inverse of a path is $\alpha^{-1}(s) = \alpha(1 - s)$.
- This is not a group, yet!

The Fundamental group II

Homotopy

Let $\alpha, \beta : I \rightarrow M$ be loops at p_0 .

α and β are *homotopic*, $\alpha \sim \beta$, if there exists a continuous map $F : I \times I \rightarrow M$ such that

- $F(s, 0) = \alpha(s)$ and $F(s, 1) = \beta(s)$ for all $s \in I$.
- $F(0, t) = F(1, t) = p_0$ for all $t \in I$.

- $\alpha \sim \beta$ is an equivalence relation.
- Let $[\alpha]$ be the equivalence class given by α .
- Define a product on equivalence classes by $[\alpha] * [\beta] = [\alpha * \beta]$.
- This gives the *fundamental group* $\pi_1(M, p_0)$.¹
- Examples: $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(\mathbb{R}^2 \setminus \{0\}) = \mathbb{Z}$, $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$.
- Note $\pi_1(M \times N) = \pi_1(M) \oplus \pi_1(N)$.

¹If M is arcwise connected then $\pi_1(M, p_0)$ is isomorphic to $\pi_1(M, p_1)$.

Higher Homotopy groups $\pi_n(M)$

- This generalizes naturally to higher homotopy groups: Consider maps from the cube $I^n = I \times \cdots \times I$ to a manifold M such that all the points on the boundary ∂I^n of the cube are mapped to $p_0 \in M$:

$$\alpha : (I^n, \partial I^n) \rightarrow (M, p_0).$$

- Again we can form the product $\alpha * \beta$ and define the equivalence classes $[\alpha]$ (also known as homotopy classes).
- This gives us the n th homotopy group $\pi_n(M)$.
- Homotopy groups are Abelian for $n > 1$, i.e. $[\alpha] * [\beta] = [\beta] * [\alpha]$.

Summary of important results

- Manifolds M with $\pi_1(M) = 1$ are called *simply-connected*.
- $\pi_n(S^n) = \mathbb{Z}$
(the integer is known as the *degree* of the map and is related to the number of pre-images)
- $\pi_n(S^d) = 1$ for $1 \leq n < d$
(contractible, not onto)
- $\pi_{n+1}(S^n) = \mathbb{Z}_2$, for $n \geq 3$, but $\pi_3(S^2) = \mathbb{Z}$ (related to Hopf bundle)
- $\pi_{n+2}(S^2) = \mathbb{Z}_2$ for $n \geq 2$.
(Homotopy groups of spheres really are complicated!)
- *Spectral sequences* are an important tool:
Let G be a Lie group with subgroup H then
$$\cdots \rightarrow \pi_n(H) \rightarrow \pi_n(G) \rightarrow \pi_n(G/H) \rightarrow \pi_{n-1}(H) \rightarrow \pi_{n-1}(G) \rightarrow \pi_{n-1}(G/H) \rightarrow \cdots$$
is a long exact sequence. (example: $G = S^3$, $H = S^1$, $G/H = S^2$)

Homotopy groups and Field Theory

- *Why are these homotopy groups important for field theories?*
- Field configurations are maps $\phi : \mathbb{R}^d \rightarrow M$, from flat space to a target space.
- Homotopies of maps occur naturally (e.g. time evolution is continuous and connects different field configurations in the same homotopy class).
- Two scenarios naturally give rise to homotopy groups. Both arise from boundary conditions (due to finite energy).
 - 1 *One-point compactification:* There is a unique vacuum $v_0 \in M$, namely, $\phi(\mathbf{x}) = v_0$ for $\mathbf{x} \rightarrow \infty$. So, we can identify all these points, so that topologically $\mathbb{R}^d \cup \{\infty\} = S^d$. So, we need

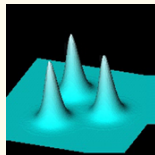
$$\pi_d(M).$$

- 2 *Nontrivial maps at infinity:* The vacuum is degenerate and forms a submanifold N of M . Then, in the limit $|\mathbf{x}| \rightarrow \infty$ there is a continuous map $\phi|_{\infty} : S_{\infty}^{d-1} \rightarrow N$. So, we need

$$\pi_{d-1}(N).$$

Classification of solitons

$\pi_n(S^k)$	ungauged	gauged
$\pi_1(S^1)$	Kinks	Vortices
$\pi_2(S^2)$	Baby-Skyrmions, Lumps	Monopoles
$\pi_3(S^3)$	Skyrmions	Instantons
$\pi_3(S^2)$	Hopf Solitons	



Ginzburg-Landau vortices

- The Ginzburg-Landau energy is given by

$$V = \frac{1}{2} \int \left(B^2 + \overline{D_i \phi} D_i \phi + \frac{\lambda}{4} (1 - \overline{\phi} \phi)^2 \right) d^2 \mathbf{x}.$$

where $\mathbf{x} = (x, y)$.

- This is invariant under

$$\begin{aligned} \phi(\mathbf{x}) &\mapsto e^{i\alpha(\mathbf{x})} \phi(\mathbf{x}) \\ a_i(\mathbf{x}) &\mapsto a_i(\mathbf{x}) + \partial_i \alpha(\mathbf{x}), \end{aligned}$$

where $e^{i\alpha(\mathbf{x})}$ is a spatially varying phase.

- Here $D_i = \partial_i \phi - ia_i \phi$ is the covariant derivative and

$$B = \partial_1 a_2 - \partial_2 a_1$$

is the magnetic field.

- The vacuum is $\phi = 1$, $a_i = 0$ and gauge transformations of this.

- Asymptotically, for finite energy fields, we can fix the gauge so that

$$\lim_{\rho \rightarrow \infty} \phi(\rho, \theta)$$

exists and varies continuously with θ , where $(x, y) = (\rho \cos \theta, \rho \sin \theta)$.

- Since $|\phi| \rightarrow 1$ as $\rho \rightarrow \infty$,

$$\lim_{\rho \rightarrow \infty} (\rho, \theta) = e^{i\alpha(\theta)},$$

where α is a continuous function of θ .

- Winding number* N : As θ increases from 0 to 2π , $\alpha(\theta)$ increases by $2\pi N$ (ϕ is single valued). N is an arbitrary integer, cannot change under smooth deformations of the field, remains constant in time.
- N is also invariant under smooth gauge transformations.

Topological charge II

- In polar coordinates (ρ, θ)

$$V = \frac{1}{2} \int_0^{\infty} \int_0^{2\pi} \left(B^2 + \overline{D_\rho \phi} D_\rho \phi + \frac{1}{\rho^2} \overline{D_\theta \phi} D_\theta \phi + \frac{\lambda}{4} (1 - \overline{\phi} \phi)^2 \right) \rho \, d\rho \, d\theta.$$

- By Stokes theorem

$$\int_{\mathbb{R}^2} B \, d^2x = \int_0^{2\pi} a_\theta \, d\theta \Big|_{\rho \rightarrow \infty}$$

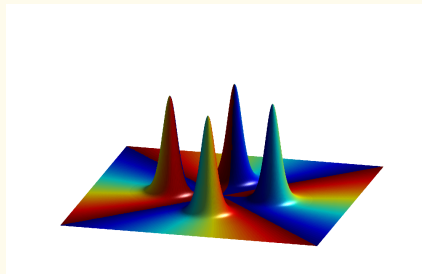
- As $\rho \rightarrow \infty$, the covariant derivative $D_\theta \phi = \partial_\theta - ia_\theta \phi$ has to vanish. Since $\phi = e^{i\alpha(\theta)}$ we have $a_\theta = \frac{d\alpha}{d\theta}$. Hence

$$\int_{\mathbb{R}^2} B \, d^2x = \alpha(2\pi) - \alpha(0) = 2\pi N.$$

so N measures the magnetic flux units in the plane.

Topological charge III

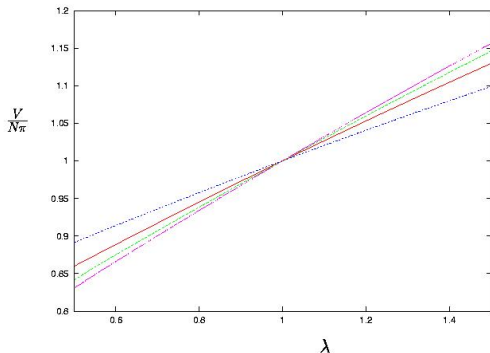
- If ϕ has only isolated zeros, then the number of these (counted with multiplicity) is N .
- A zero of ϕ is said to have multiplicity k , if on a small circle enclosing the zero, $-\arg \phi$ increases by $2\pi k$. For simple zeros $k = \pm 1$.



Energy of N Ginzburg-Landau vortices

- Let E_N be the minimal energy V of N vortices.

$\lambda < 1$ $E_N < NE_1$ the vortices attract (Type I)
 $\lambda > 1$ $E_N > NE_1$ the vortices repel (Type II)
 $\lambda = 1$ $E_N = NE_1$ no forces between static vortices



Vortices at critical coupling $\lambda = 1$

- By “completing the square” V can be written as

$$V = \frac{1}{2} \int \left(\left(B - \frac{1}{2} (1 - \bar{\phi}\phi) \right)^2 + (\overline{D_1\phi + iD_2\phi}) (D_1\phi + iD_2\phi) + B \right) d^2x.$$

- Recall that

$$\int B d^2x = 2\pi N, \quad \text{so} \quad V \geq \pi N.$$

- Bogomolny equations:

$$\begin{aligned} D_1\phi + iD_2\phi &= 0 \\ B - \frac{1}{2} (1 - \bar{\phi}\phi) &= 0. \end{aligned}$$

- These equations cannot be solved analytically. However, a lot is known about the solutions.

The Vortex moduli space

- For given topological charge N , the Bogomolny equations have a $2N$ dimensional manifold of static solutions, known as the *moduli space* M_N . (Gauge equivalent solutions are identified.)
- All zeros of ϕ have positive multiplicity (generically there are only simple zeros).
- A solution is completely determined by the locations of these zeros, which can be anywhere. N unordered points in \mathbb{R}^2 require $2N$ coordinates.
- There are no *static* forces between vortices for $\lambda = 1$, however, there will be *velocity dependent* forces.

Relativistic vortex dynamics

- The standard relativistic Lagrangian is

$$\mathcal{L} = \frac{1}{2} \overline{D_\mu \phi} D^\mu \phi - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{\lambda}{8} (1 - \overline{\phi} \phi)^2,$$

where $x^\mu = (t, \mathbf{x})$.

- In the following, we will often use complex coordinates $z = x + iy$.
- We can parametrize the moduli space for $\lambda = 1$ in terms of the vortex positions Z_j . Assuming that Z_j are time dependent gives rise to the reduced Lagrangian

$$L_{\text{red.}} = \frac{1}{2} \sum_{r,s=1}^N \left(g_{rs} \dot{Z}_r \dot{Z}_s + g_{r\bar{s}} \dot{Z}_r \dot{\bar{Z}}_s + g_{\bar{r}s} \dot{\bar{Z}}_r \dot{Z}_s \right) - V_{\text{red.}},$$

where

$$V_{\text{red.}} = \frac{\lambda - 1}{8} \int (1 - \overline{\phi} \phi)^2 d^2x.$$

Properties of the moduli space

- Setting $h = \log |\phi|^2$ the Bogomolny equations imply

$$\nabla^2 h + 1 - e^h = 4\pi \sum_{r=1}^N \delta^2(z - Z_r).$$

- The δ functions arise because h has logarithmic singularities at the zeros Z_r of ϕ .
- Expanding h around the point Z_r gives

$$h(z, \bar{z}) = 2 \log |z - Z_r| + a_r + \frac{1}{2} \bar{b}_r (z - Z_r) + \frac{1}{2} b_r (\bar{z} - \bar{Z}_r) + \dots$$

- After a long calculation

$$L_{\text{red.}} = \frac{\pi}{2} \sum_{r,s=1}^N \left(\delta_{rs} + 2 \frac{\partial b_s}{\partial Z_r} \right) \dot{Z}_r \dot{\bar{Z}}_s - V_{\text{red.}}$$

The metric on the moduli space

- The moduli space metric

$$g = \frac{\pi}{2} \sum_{r,s=1}^N \left(\delta_{rs} + 2 \frac{\partial b_s}{\partial Z_r} \right) dZ_r d\bar{Z}_s$$

is Kähler.

- This structure provides a lot of information about the metric, although it is only known implicitly.
- The moduli space approximation captures the dynamics of vortices, in particular right-angle scattering.

- The Schrödinger-Chern-Simons Lagrangian

$$\begin{aligned}\mathcal{L}_{SCS} = & \frac{i}{2} (\bar{\phi} D_0 \phi - \phi \overline{D_0 \phi}) + B a_0 + e_1 a_2 - e_2 a_1 - a_0 \\ & - \frac{1}{2} B^2 - \frac{1}{2} \overline{D_i \phi} D_i \phi - \frac{\lambda}{8} (1 - \bar{\phi} \phi)^2,\end{aligned}$$

is a model for vortex dynamics in superconductors.

- This Lagrangian is gauge invariant and Galilean invariant.
- \mathcal{L}_{SCS} give rise to first order vortex dynamics.
- For λ close to one, we can again use our moduli space M_N to approximate the dynamics of N vortices.

Moduli approximation and the Kähler potential

- Now, the reduced Lagrangian is also first order

$$L_{\text{red.}} = - \sum_{i=1}^{2N} \mathcal{A}_i(\mathbf{y}) \dot{y}_i - V_{\text{red.}}(\mathbf{y}),$$

where \mathbf{y} are the coordinates on the moduli space and

$$V_{\text{red.}} = \frac{\lambda - 1}{8} \int (1 - \bar{\phi}\phi)^2 d^2x.$$

- \mathcal{A} is a gauge potential, and $\mathcal{F} = d\mathcal{A}$ the corresponding field strength.
- The equations of motion are

$$\mathcal{F}_{ij} \dot{y}_j = - \frac{\partial V_{\text{red.}}}{\partial y_i}.$$

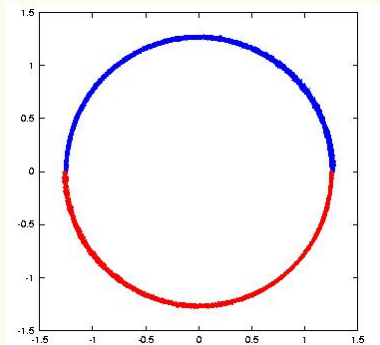
- The field strength \mathcal{F} is

$$\mathcal{F} = -i\pi \sum_{r,s=1}^N \left(\delta_{rs} + 2 \frac{\partial b_s}{\partial Z_r} \right) dZ_r \wedge d\bar{Z}_s$$

which is the Kähler form associated to the metric g on M_N .

Moduli space approximation

- For λ close to 1, two vortices circle around each other anticlockwise.
- Moduli space approximation is in agreement with numerical simulation.



Vortices on various domains

- We can consider physical spaces with a different metric, e.g.

$$ds^2 = dt^2 - \Omega(x, y)(dx^2 + dy^2),$$

where Ω is a Riemannian metric on a physical space X .

- Again we can “complete the square” and obtain the Bogomolny equations

$$\begin{aligned}D_1\phi + iD_2\phi &= 0 \\ B - \frac{\Omega}{2}(1 - \bar{\phi}\phi) &= 0,\end{aligned}$$

where $B = f_{12}$.

- The integral

$$c_1 = \frac{1}{2\pi} \int_X f = \frac{1}{2\pi} \int_X B \, d^2x$$

is an integer. This topological invariant is known as the first *Chern number*.

Compact domains and the Bradlow limit

- We can integrate the second Bogomolny equation over X and obtain

$$2 \int_X B \, d^2x + \int_X |\phi|^2 \Omega \, d^2x = \int_X \Omega \, d^2x.$$

- If X has a finite area A we obtain

$$4\pi N + \int_X |\phi|^2 \Omega \, d^2x = A.$$

- This gives us the Bradlow limit

$$A \geq 4\pi N$$

in other words, a vortex needs at least an area of 4π .

- At the Bradlow bound $A = 4\pi N$ both equations can trivially be solved by $\phi = 0$ and $B = \frac{\Omega}{2}$.
- For the torus T^2 the moduli space metric has been calculated as an expansion around the Bradlow limit.

Hyperbolic vortices

- Setting $h = \log |\phi|^2$ we can again derive an equation for h :

$$\nabla^2 h + \Omega - \Omega e^h = 4\pi \sum_{r=1}^N \delta^2(z - Z_r).$$

- For hyperbolic space

$$ds^2 = \frac{8}{(1 - |z|^2)^2} dz d\bar{z}$$

with $|z| < 1$, the equation can be transformed to Liouville's equation, which is integrable.

- In this case, the moduli space is known explicitly, and

$$\phi = \frac{1 - |z|^2}{1 - |f|^2} \frac{df}{dz}.$$

$f(z)$ has the rather simple form

$$f(z) = \prod_{i=1}^{N+1} \left(\frac{z - c_i}{1 - \bar{c}_i z} \right)$$

where $|c_i| < 1$. The positions of the vortices are the zeros of $\frac{df}{dz}$.

Metric for Hyperbolic vortices

- In hyperbolic space, the metric is

$$g = \frac{\pi}{2} \sum_{r,s=1}^N \left(\Omega(Z_r) \delta_{rs} + 2 \frac{\partial b_s}{\partial Z_r} \right) dZ_r d\bar{Z}_s$$

but now we can calculate b_s for special cases.

- The metric for n vortices on a regular polygon with m vortices fixed at the origin is given by

$$ds^2 = \frac{4\pi n^3 |\alpha|^{2n-2} d\alpha d\bar{\alpha}}{(1 - |\alpha|^{2n})^2} \left(1 + \frac{2n(1 + |\alpha|^{2n})}{\sqrt{(m+1)^2(1 - |\alpha|^{2n})^2 + 4n^2|\alpha|^{2n}}} \right)$$

for $n \neq m+1$, and by

$$ds^2 = \frac{12\pi n^3 |\alpha|^{2n-2} d\alpha d\bar{\alpha}}{(1 - |\alpha|^{2n})^2}$$

for $m+1 = n$. The nontrivial zeros are at $z = \alpha e^{2\pi ik/n}$ for $k = 0, \dots, n-1$.

- Kinetic and potential energy (with $z = x + iy$):

$$T = \frac{1}{2} \int \frac{4|W_t|^2}{(1 + |W|^2)^2} \frac{4dx dy}{(1 + |z|^2)^2}, \quad V = \frac{1}{2} \int \frac{4(|W_x|^2 + |W_y|^2)}{(1 + |W|^2)^2} dx dy.$$

- Static minimal energy solutions are rational maps of degree n

$$W = \frac{a_1 z^n + \dots + a_{n+1}}{a_{n+2} z^n + \dots + a_{2n+2}},$$

with potential energy $V = 4\pi n$.

- Make moduli space coordinates time dependent

$$W(t, z) = \frac{z^n + q_2(t)z^{n-1} + \dots + q_{n+1}(t)}{q_{n+2}(t)z^n + q_{n+3}(t)z^{n-1} + \dots + q_{2n+2}(t)}$$

- Then the metric is given by

$$T = \frac{1}{2} \sum_{ij} \gamma_{ij} \dot{q}_i \bar{\dot{q}}_j, \quad \gamma_{ij} = \int \frac{4}{(1 + |W|^2)^2} \frac{\partial W}{\partial q_i} \frac{\partial \bar{W}}{\partial q_j} \frac{4dx dy}{(1 + |z|^2)^2}.$$

Charge 1 lumps

- Use projective equivalence class $[L]$ of $GL(2, \mathbb{C})$ matrices,

$$\frac{a_1 z + a_2}{a_3 z + a_4} \leftrightarrow \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \right].$$

- $SO(3) \times SO(3)$ symmetry

$$([U_1], [U_2]) : [L] \mapsto [U_1 L U_2^{-1}].$$

- Unique polar decomposition:

$$[L] = [U(\Lambda I_2 + \boldsymbol{\lambda} \cdot \boldsymbol{\tau})],$$

where $([U], \boldsymbol{\lambda}) \in PU(2) \times \mathbb{R}^3$, $\Lambda = \sqrt{1 + \boldsymbol{\lambda}^2}$, $\lambda = |\boldsymbol{\lambda}|$, and τ_1, τ_2, τ_3 are the Pauli spin matrices.

Charge 1 lumps continued

- Let γ be an $SO(3) \times SO(3)$ invariant Kähler metric on Rat_1 . Then

$$\gamma = A_1 d\lambda \cdot d\lambda + A_2(\lambda \cdot d\lambda)^2 + A_3 \sigma \cdot \sigma + A_4(\lambda \cdot \sigma)^2 + A_1 \lambda \cdot (\sigma \times d\lambda),$$

where A_1, \dots, A_4 are smooth functions of λ only, all determined from the single function $A_1 = A(\lambda)$ by the relations

$$A_2 = \frac{A(\lambda)}{1 + \lambda^2} + \frac{A'(\lambda)}{\lambda}, \quad A_3 = \frac{1}{4}(1 + 2\lambda^2)A(\lambda), \quad A_4 = \frac{1}{4\lambda}(1 + \lambda^2)A'(\lambda).$$

Here $\sigma_1, \sigma_2, \sigma_3$ are the left invariant one forms.

- For the L^2 metric, one finds that

$$A = \frac{32\pi\mu[\mu^4 - 4\mu^2 \log \mu - 1]}{(\mu^2 - 1)^3}.$$

where $\mu = \frac{\Lambda + \lambda}{\Lambda - \lambda}$.