Measure Theory: Exercises 4

1. Show that if $f : \mathbf{R} \to \mathbf{R}$ is differentiable everywhere on $\mathbf{R}$, this its derivative f' is Borel measurable.

2. For every $i = 1, 2, \ldots$ define the function $f_i : [0,1] \to \{0,1\}$ by $f_i(t) = 1$ if the ith digit in the decimal representation of t is odd and $f_i(t) = 0$ if the ith digit in the decimial representation of t is even. For example, if $x = .74182....$ then $f_1(x) = 1$, $f_2(x) = 0$, $f_3(x) = 1$, and so on. Show that there cannot exist a measure function $f : [0,1] \to \{0,1\}$ such that $\lim_{i\to\infty} f_i = f$ a.e.

3.Let f, g be continuous real-valued functions defined on $\mathbf{R}$. Show that if $f = g$ λ-almost everywhere then $f = g$ everywhere.

4. Consider the function $g(x) = 0$ if $x = 0$ or x is irrational, and $g(x) = \frac{1}{q}$ if $x = \frac{p}{q}$ and p, q are relatively prime and $q > 0$. Show that g is continuous λ-almost everywhere.