# Measure Theory Second Week

**Outer Measures:**

Let $X$ be a set,

$\mathcal{P}(X)$ the collection of all subsets of $X$.

An outer measure $\mu : \mathcal{P}(X) \to [0, \infty]$ (defined on all subsets) is a function such that

(a) $\mu(\emptyset) = 0$,

(b) if $A \subseteq B$ then $\mu(A) \leq \mu(B)$,

(c) if $A_1, A_2, \ldots$ is a sequence of subsets then

$\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$ (subadditive).

For outer measures $\mu$ that are not measures there is some sequence $A_1, A_2, \ldots$ of disjoint sets such that

$\sum_{i=1}^{\infty} \mu(A_i) > \mu(\cup_{i=1}^{\infty} A_i)$.

For finitely additive measures $\mu$ that are not measures there is some sequence $A_1, A_2, \ldots$ of disjoint sets such that

$\sum_{i=1}^{\infty} \mu(A_i) < \mu(\cup_{i=1}^{\infty} A_i)$.

In general, outer measures are not measures,

as they are defined on all subsets;

usually measures require some restriction to a collection of measurable subsets.

**Examples:**

(a) $\mu(A) = 0$ if $A = \emptyset$ and

$\mu(A) = 1$ if $A \neq \emptyset$.

(b) $\mu(A) = 0$ if $A$ is countable and

$\mu(A) = 1$ if $A$ is uncountable.

(c) Let $(X, \mathcal{A}, \mu)$ be a measurable space. Define $\mu^*(B) = \inf_{A \in \mathcal{A},\ A \supset B} \ \mu(A)$.

**Lebesgue outer measure:**

$\lambda^*$ is defined on all subsets of $\mathbf{R}$.

$\lambda^*(A) =$

$\inf\{\sum_{i=1}^{\infty} b_i - a_i \mid \cup_{i=1}^{\infty} (a_i, b_i) \supset A\}$.

**Lemma (1.3.2):** Lebesgue outer measure is an outer measure and assigns to every interval its length.

**Proof:** The empty set is covered by any collection of open intervals, hence also of lengths $\epsilon/2, \epsilon/4, \ldots,$

therefore $\lambda^*(\emptyset) = 0$.

If $A \subseteq B$ then any collection of intervals covering $B$ also covers $A$.

Hence the collection of coverings for $A$ involves a larger collection than that for $B$,

and therefore $\lambda^*(A) \leq \lambda^*(B)$.

Let $\epsilon > 0$ be given. Any covering collection used to define $\mu(A_i)$ to within $\frac{\epsilon}{2^i}$ also is a covering collection for $\cup_i A_i$.

Hence after taking the infinum on all coverings of $\cup_i A_i$ and ignoring the $\epsilon$

it follows that $\lambda^*(\cup_i A_i) \leq \sum_i \lambda^*(A_i)$.

Finally, letting $I$ be any interval from $a$ to $b$ with $b > a$, be in closed, open, or open on one end and closed on the other,

the sequence $(a-\epsilon, b+\epsilon)$ covers the interval,

and so $\lambda^*$ of the interval is no more than $b - a$.

One the other hand, it suffices to show that $\lambda^*$ of the closed interval $[a, b]$ is at least $b-a$.

Because it is compact, any collection of covering open intervals can be reduced to a finite covering collection.

Now easy to show that if the lengths of this finite cover did not add up to at least $b - a$ they could not reach from $a$ to $b$.

**Definition:** Let $\mu$ be a outer measure on $X$. A subset $B$ is $\mu$-measurable if for every subset $A$ of $X$ it holds that

$$\mu(A) = \mu(A \cap B) + \mu(A \backslash B).$$

Subadditivity of outer meaures implies already that $\mu(A) \leq \mu(A \cap B) + \mu(A \backslash B)$,

so whenever $\mu(A) = \infty$ it is automatically true.

A *Lebesgue* measurable set is one that is measurable with respect to Lebesgue outer measure,

and the measure $\lambda$ is the measure $\lambda^*$ restricted to the Lebesgue measurable sets.

**Lemma: (1.3.5)** Let $\mu$ be an outer measure on $X$. Every subset $B$ such that $\mu(B) = 0$ or $\mu(X \backslash B) = 0$ is $\mu$-measurable.

**Proof:** We need only show for every subset $A$ that $\mu(A) \geq \mu(A \cap B) + \mu(A \cap (X \backslash B))$.

With $\mu(B) = 0$ or $\mu(X \backslash B) = 0$ it follows by monotonicity.

If $\mu$ is an outer measure,

let $\mathcal{M}_\mu$ be the collection of $\mu$ measurable sets.

**Theorem (1.3.6):**

$\mathcal{M}_\mu$ is a sigma-algebra and

$\mu$ is a measure on $\mathcal{M}_\mu$.

**Proof:** From the previous lemma and the definintion of $\mathcal{M}_\mu$, $X$ is in $\mathcal{M}_\mu$, $\mu(\emptyset) = 0$, and $A \in \mathcal{M}_\mu$ if and only if $X \backslash A \in \mathcal{M}_\mu$.

Next we show that $\mathcal{M}_\mu$ is an algebra and it is finitely additive.

Let $B_1, B_2 \in \mathcal{M}_\mu$; with closure by complementation already demonstrated, it suffices to show that $B_1 \cap B_2$ is also in $\mathcal{M}_\mu$.

Let $A$ be any subset: as $B_2$ is in $\mathcal{M}_\mu$

$\mu(A \cap B_1) =$

$\mu(A \cap B_1 \cap B_2) + \mu((A \cap B_1) \backslash B_2)$ and

$\mu(A \backslash B_1) = \mu(A \cap (X \backslash B_1)) =$

$\mu((A \backslash B_1) \cap B_2) + \mu((A \backslash B_1) \backslash B_2)$.

As $B_1 \in \mathcal{M}_\mu$: $\mu(A) = \mu(A \cap B_1) + \mu(A \backslash B_1)$

$\mu(A) = \mu(A \cap B_1 \cap B_2) + \mu((A \cap B_1) \backslash B_2) + \mu((A \backslash B_1)) \geq$

$\mu(A \cap B_1 \cap B_2) + \mu(A \backslash (B_1 \cap B_2)) \geq \mu(A)$,

(by subadditivity)

hence $B_1 \cap B_2$ is also in $\mathcal{M}_\mu$.

Furthermore, assuming $B_1, B_2 \in \mathcal{M}_\mu$ are disjoint,

and letting $A = B_1 \cup B_2$ be the set chosen,

we have $A \backslash B_1 = B_2$, $A \cap B_1 = B_1$

and $\mu(A) = \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2)$.

Therefore $\mu$ is finitely additive on $\mathcal{M}_\mu$.

Let $B_1, B_2, \ldots$ be an infinite sequence of mutually disjoint members of $\mathcal{M}_\mu$ and let $A$ be any subset:

It follows from finite additivity and induction that

$\mu(A) = \sum_{i=1}^{n} \mu(A \cap B_i) + \mu(A \backslash (\cup_{i=1}^{n} B_i).$

Letting $n$ go to infinity,

$\mu(A) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A \cap B_i) + \lim_{n\to\infty} \mu(A \backslash (\cup_{i=1}^{n} B_i).$

By monotonicity $\mu(A \backslash (\cup_{i=1}^{\infty} B_i)) \leq \lim_{n\to\infty} \mu(A \backslash (\cup_{i=1}^{n} B_i))$

and by the definition of infinite sums

$\lim_{n\to\infty} \sum_{i=1}^{n} \mu(A \cap B_i) = \sum_{i=1}^{\infty} \mu(A \cap B_i),$

so $\mu(A) \geq \sum_{i=1}^{\infty} \mu(A \cap B_i) + \mu(A \backslash (\cup_{i=1}^{\infty} B_i))$

Therefore by the above and $\mu$ being an outer measure, $\mu(A) \geq$

$$\sum_{i=1}^{\infty} \mu(A \cap B_i) + \mu(A \backslash (\cup_{i=1}^{\infty} B_i)) \geq$$

$$\mu(A \cap (\cup_{i=1}^{\infty} B_i)) + \mu(A \backslash (\cup_{i=1}^{\infty} B_i)) \geq \mu(A).$$

It follows that $\cup_{i=1}^{\infty} B_i$ is in $\mathcal{M}_\mu$.

Sigma-additivity on the disjoint sequence of the $B_i$ follows from setting $A = \cup_{i=1}^{\infty} B_i$.

Starting from any sequence $A_1, \ldots$ of sets in $\mathcal{M}_\mu$, by chosing the disjoint $B_i = A_i \backslash (\cup_{j=0}^{i-1} A_j)$ (with $A_0 = \emptyset$) we get $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$ in $\mathcal{M}_\mu$. $\square$

**Lemma:** Every Borel subset of $\mathbf{R}$ is Lebesgue measurable.

**Proof:** Given that the Lebesgue measurable sets define a sigma algebra,

and the Borel subsets are the smallest sigma algebra containing intervals of the form $I = (-\infty, c]$, given any subset $A$ we need that

$\lambda^*(A) = \lambda^*(A \cap I) + \lambda^*(A \backslash I)$.

We can break the $i$th open interval $(a_i, b_i)$ covering $A$ into two inteverals, $(a_i, c + \frac{\epsilon}{2^i})$ and $(c, b_i)$ whenever $a_i < c < b_i$.

In this way we cover both $A \cap I$ and $A \backslash I$ and show that $\lambda^*(A \cap I) + \lambda^*(A \backslash I) \leq \lambda^*(A) + \epsilon$ for every $\epsilon > 0$;

together with subadditivity, the equality follows.

**More on Lebesgue measure:**

**Lemma (regularity):** Let $B$ be a Lebesgue measurable subset of finite measure.

For every $\epsilon > 0$ there is an open set $A$ and a compact set $C$ such that $C \subseteq B \subseteq A$

and $\lambda(A \backslash C) < \epsilon$.

**Proof:**

As the measure $\lambda(B)$ is approximated by open covers,

there is an open cover of $B$ whose union $A$ has measure less than $\lambda(B) + \epsilon/3$

By sigma additivity,

there is an $n$ large enough so that

$\lambda(B \cap [-n, n]) > \lambda(B) - \epsilon/3.$

Cover $[-n, n] \setminus B$ with an open set $G$ so that $\lambda(G) > \lambda([-n, n] \setminus B) + \epsilon/3.$

$C = [-n, n] \setminus G$ is a closed set contained in $B$ whose measure is more than $\lambda(B) - 2\epsilon/3.$

**Lemma:** Lebesgue measure is translation invariant,

meaing that for any given $r \in R$,

a set $A$ is Lebesgue measurable

if and only if $A + r := \{a + r \mid a \in A\}$ is Lebesgue measurable

and $\lambda^*(A) = \lambda^*(A + r)$.

**Proof:** Let $(I_i \mid i = 1, 2, \dots)$ be a collection of open intervals covering $A$.

The intervals $(I_i + r)$ cover $A + r$ and each interval has the same length.

This shows that $\lambda^*(A + r) \leq \lambda^*(A)$,

and the same arguement shifting by $-r$ shows the opposite inequality.

Likewise the intersection property with any subset of $\mathbf{R}$ that confirms that $A$ and $X \setminus A$ are Lebesgue measurable

shows the same for $A + r$ and $(X \setminus A) + r$ after all sets are shifted by $r$.

**Theorem:** Given the axiom of choice,

there is a subset of $[0, 1)$ that is not Lebesgue measurable.

**Proof:** Define an equivalence relation on $r, s \in [0, 1)$

by $r \sim s \Leftrightarrow r - s$ is rational.

Define addition modulo 1,

so that $b + c$ is $b + c - 1$ if $b + c \geq 1$.

List the rational numbers $a_1, a_2, \ldots$ in $[0, 1)$.

Let $B$ be a set of representatives for the equivalence relation (Axiom of Choice)

meaning that $B$ intersects every equivalence class one and only once,

or that for every $r \in [0, 1)$ there is one and only one $i$ with $r + a_i \in B$.

This means that $\cup_{i=1}^{\infty}(B - a_i)$ partitions $[0, 1)$:

for every $r$ there is some $b \in B$ and $a_i$ such that $r = b - a_i$

and if $r \in (B - a_i) \cap (B - a_j) \neq \emptyset$ for distinct $a_i \neq a_j$

then $r = b_i - a_i = b_j - a_j$ for some $b_i, b_j \in B$ and the equivalence relation sharing both $r + a_i$ and $r + a_j$ have two representatives $b_i$ and $b_j$ in $B$, a contradiction.

Assume that $B$ is Lesbegue measurable.

Notice that translation invariance holds also in the modulo arithmetic,

due to a secondary shift of the measurable subset that went over the value of 1.

So every $B + a_i$ must be Lebesgue measurable and have the same measure.

This measure can neither be 0 or anything positive,

as that would imply that the whole set $[0, 1)$ is either infinite in measure or zero in measure,

when it is really of measure one.