LTCC Mathematical Biology - Mock

Question 1.

(a) Consider the motion of non-interacting particles along the x axis. Each particle i moves a distance dx_i every time step, which lasts τ seconds. At time $t = 0$, all particles are at $x = 0$. Let N be the total number of particles, $x_i(n)$ be the position of particle i after n time steps, and $X(n) = \frac{1}{N}\Sigma_{i=1}^{N} x_i(n)$ be the average position of the particles after n time steps (i.e. at time t=$n\tau$).

i. Assume that every τ seconds, each particle i either a) moves a distance $dx_i = \delta$ with probability 1/3, b) moves a distance $dx_i = -\delta$ with probability 1/3, or c) does not move ($dx_i = 0$) with probability 1/3. Calculate the mean displacement of the particles $E(X(n))$ after n steps.

Solution: The mean displacement of the particles after n steps is

$$\begin{aligned}
\langle X(n) \rangle &= \frac{1}{N} \sum_{i=1}^{N} \langle x_i(n) \rangle \\
&= \frac{1}{N} \sum_{i=1}^{N} \left[\langle x_i(n-1) \rangle + \langle dx_i \rangle \right] \\
&= \frac{1}{N} \sum_{i=1}^{N} \left[\langle x_i(n-1) \rangle + \delta \cdot \frac{1}{3} + (-\delta) \cdot \frac{1}{3} \right] \\
&= \frac{1}{N} \sum_{i=1}^{N} \langle x_i(n-1) \rangle \\
&= \langle X(n-1) \rangle. \qquad (1)
\end{aligned}$$

Thus particles go nowhere on average and $E(X(n)) = 0$ from the initial condition.

ii. Assume that every τ seconds, each particle i either a) moves a distance $dx_i = \delta$ with probability 1/4, or b) moves a distance $dx_i = -\delta$ with probability 3/4. Calculate the mean displacement of the particles $E(X(n))$ after n steps. Write your answer in terms of n, δ and τ.

Solution: The mean displacement of the particles after n steps is

$$\begin{aligned}\langle X(n)\rangle &= \frac{1}{N}\sum_{i=1}^{N}\langle x_i(n)\rangle \\ &= \frac{1}{N}\sum_{i=1}^{N}\left[\langle x_i(n-1)\rangle + \langle dx_i\rangle\right] \\ &= \frac{1}{N}\sum_{i=1}^{N}\left[\langle x_i(n-1)\rangle + \delta\cdot\frac{1}{4} + (-\delta)\cdot\frac{3}{4}\right] \\ &= \frac{1}{N}\sum_{i=1}^{N}\langle x_i(n-1)\rangle - \frac{\delta}{2} \\ &= \langle X(n-1)\rangle - \frac{\delta}{2}. \end{aligned} \tag{2}$$

Repeated n times and applying the initial condition gives $E(X(n)) = -n\delta/2$.

As before, assume that every τ seconds, each particle i either a) moves a distance $dx_i = \delta$ with probability 1/4, or b) moves a distance $dx_i = -\delta$ with probability 3/4. Let P(x,t) be the probability of finding a particle at point x at time t, with $p(x,t)$ the associated probability density.

iii. Write $p(x, t+\tau)$ in terms of $p(x-\delta, t)$ and $p(x+\delta, t)$.
Solution:

$$p(x, t+\tau) = \frac{3}{4}p(x+\delta, t) + \frac{1}{4}p(x-\delta, t) \tag{3}$$

iv. Take your answer from part iii and Taylor expand each side with respect to the appropriate variable to write the equivalent of the diffusion equation in the limit $\delta \to 0$ and $\tau \to 0$ in this case.
Solution: Taylor expanding both sides with respect to the appropriate variable, and canceling terms, gives,

$$\begin{aligned}\tau\frac{\partial p}{\partial t} &= \left(-\frac{1}{4}+\frac{3}{4}\right)\delta\frac{\partial p}{\partial x} + \frac{\delta^2}{2}\frac{\partial^2 p}{\partial x^2}, \\ \frac{\partial p}{\partial t} &= \frac{\delta}{2\tau}\frac{\partial p}{\partial x} + \frac{\delta^2}{2\tau}\frac{\partial^2 p}{\partial x^2},\end{aligned}$$

in the limit $\delta \to 0$ and $\tau \to 0$.

(b) Consider the diffusion equation for the concentration $n(x,t)$ of a cloud of particles

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2},$$

where $D > 0$. Consider the homogeneous steady solution $n_0 =$ constant. Insert the wavelike perturbation

$$n(x,t) = n_0 + \tilde{n}(x,t), \quad \text{where } \tilde{n} = \epsilon e^{\sigma t + ikx}$$

into the diffusion equation and derive the dispersion relation $\sigma(k)$. What does this tell you about the linear stability of n_0?

Solution: Inserting this perturbation ansatz into the diffusion equation gives the dispersion relation

$$\sigma(k) = -Dk^2 \leq 0,$$

signalling that n_0 is a stable solution, because all modes with $k > 0$ become exponentially damped.