LTCC Examination Paper, 2022-23

1. Let X be the set $\{\frac{k}{2^i} \mid i, k$ are integers and $k \neq 0\}$. For every integer i let $A_i = \{\frac{k}{2^i} \mid k$ is odd.$\}$. Let $\mathcal{A} = \{A_i | i = 1, 2, \dots\}$. What is $\sigma(\mathcal{A})$, the smallest sigma-algebra containing $\mathcal{A}$?

2. Let $A_1, A_2, \dots$ be Lebesgue measurable sets with $\sum_{i=1}^{\infty} \lambda(A_i) < \infty$. Show that $\lambda(\cap_{i=1}^{\infty} A_i) = 0$.

3. True or false: if $A_1, A_2, \dots$ is a sequence of Lebesgue measurable sets with $\sum_{i=1}^{\infty} \lambda(A_i) = \infty$ with $A_i \subseteq [0, 1]$ for every i, then $\lambda(\cap_{i=1}^{\infty} A_i) > 0$. Justify your answer.

4. A real valued function f defined on a metric space is lower-semi-continuous if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(y, x) < \delta$ implies that $f(y) < f(x) + \epsilon$, where d is the distance function. Let f be a Lebesgue measurable function defined on $[0, 1]$ with values in $[0, 1]$. Show that for every $\epsilon > 0$ there is a lower-semi-continuous function g such that $g \geq f$ and $\int_0^1 (g - f) d\lambda < \epsilon$.

5. Let A_0 and A_1 be two subsets of $\mathbf{R}$ such that $A_0 \cap A_1 = \emptyset$, $A_0 + A_0 \subseteq A_0$, $A_1 + A_1 \subseteq A_0$ and $A_0 + A_1 \subseteq A_1$. Furthermore assume that both A_i are origin symmetric, meaning that $x \in A_i$ if and only if $-x \in A_i$. Show that either A_1 is Lebesgue measurable and $\lambda(A_1) = 0$ or A_1 is not Lebesgue measurable.

6. Let f_1, f_2 be two bounded, real valued, and Lebesgue measurable functions defined on $[0, 1]$. For all $x \in [0, 1]$ let

$g(x)$ be a function defined by $f(x) = \sup_{\alpha,y_1,y_2}[\alpha f_1(y_1) + (1-\alpha) f_2(y_2)]$ under the condition that $\alpha y_1 + (1-\alpha)y_2 = x$ for $0 \le \alpha \le 1$ and $y_1, y_2 \in [0,1]$. Show that g is Lebesgue measurable.

7. Assume that the function $f : \mathbf{R} \to \mathbf{R}$ is measurable. On the set where the second derivative f'' is well defined show that it is Lebesgue measurable.