LTCC Examination Paper, exam and solutions 2022-23

1. Let X be the set $\{\frac{k}{2^i} \mid i, k$ are integers and $k \neq 0\}$. For every integer i let $A_i = \{\frac{k}{2^i} \mid k$ is odd.$\}$. Let $\mathcal{A} = \{A_i | i = 1, 2, \dots\}$. What is $\sigma(\mathcal{A})$, the smallest sigma-algebra containing $\mathcal{A}$?

The answer is simply all the subsets of $\mathcal{A}$. This is because for any $i \neq j$ the intersection $A_i \cap A_j$ is empty and $X \backslash A_i = \cup_{j \neq i} A_j$.

2.*Let $A_1, A_2, \dots$ be Lebesgue measurable sets with $\sum_{i=1}^{\infty} \lambda(A_i) < \infty$. Show that $\lambda(\cap_{i=1}^{\infty} A_i) = 0$.*

Write $f_i(x) = 1_{A_i}(x)$ for every i. The function $f = \sum_{i=1}^{\infty} f_i$ is the monotone limit of Lebesgue measurable functions $f_1 + f_2 + \cdots + f_n$ whose integration is the limit of the $\sum_{i=1}^{n} \lambda(A_i)$, which is finite. If $\lambda(\cap_{i=1}^{\infty} A_i)$ were not 0, the integration of f would be infinite, as there would be a subset of positive measure where the function is infinite.

3. *True or false: if $A_1, A_2, \dots$ is a sequence of Lebesgue measurable sets with $\sum_{i=1}^{\infty} \lambda(A_i) = \infty$ with $A_i \subseteq [0, 1]$ for every i, then $\lambda(\cap_{i=1}^{\infty} A_i) > 0$. Justify your answer.*

The answer is false. Let $A_i = [0, \frac{1}{i}]$. The infinite sum of its Lebesgue measures is infinite, and yet the infinite intersection is only the point $\{0\}$.

4. *A real valued function f defined on a metric space is lower-semi-continuous if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(y, x) < \delta$ implies that $f(y) < f(x) + \epsilon$, where d is the distance function. Let f be a Lebesgue measurable*

function defined on $[0,1]$ with values in $[0,1]$. Show that for every $\epsilon > 0$ there is a lower-semi-continuous function g such that $g \geq f$ and $\int_0^1 (g-f)d\lambda < \epsilon$.

Let $h > 0$ be a simple and Lebesgue measurable function that approximates f from above (possible because f is bounded above and below). It suffices to show the same for h. For every value r_i obtained by the function h, find an open set A_i that covers $\{x \mid h(x) = r_i\}$ with $\lambda(A_i) < \lambda(\{x \mid h(x) = r_i\}) + \frac{r_i \epsilon}{2^i}$. Now letting $g = \sum_i r_i 1_{A_i}$ we have our desired function.

5. Let A_0 and A_1 be two subsets of $\mathbf{R}$ such that $A_0 \cap A_1 = \emptyset$, $A_0 + A_0 \subseteq A_0$, $A_1 + A_1 \subseteq A_0$ and $A_0 + A_1 \subseteq A_1$. Furthermore assume that both A_i are origin symmetric, meaning that $x \in A_i$ if and only if $-x \in A_i$. Show that either A_1 is Lebesgue measurable and $\lambda(A_1) = 0$ or A_1 is not Lebesgue measurable.

Suppose for the sake of contradiction that A_1 is Lebesgue measurable and $\lambda(A_1) > 0$. As A_1 is a symmetric set about 0, it follows that $A_1 - A_1 \subseteq A_0$ contains an interval about 0. But this means through repetitive addition that A_0 must contain all of $\mathbf{R}$, a contradiction.

6. Let f_1, f_2 be two bounded, real valued, and Lebesgue measurable functions defined on $[0,1]$. For all $x \in [0,1]$ let $g(x)$ be a function defined by $f(x) = \sup_{\alpha, y_1, y_2} [\alpha f_1(y_1) + (1-\alpha) f_2(y_2)]$ under the condition that $\alpha y_1 + (1-\alpha) y_2 = x$ for $0 \leq \alpha \leq 1$ and $y_1, y_2 \in [0,1]$. Show that g is Lebesgue measurable.

Both functions f_1 and f_2 can be approximated by simple

Lebesgue measurable functions from below. Given ϵ, let f^i_ϵ be two simple functions with finitely many sets A^i_j taking on the discrete values r^i_j for $i = 1, 2$ approximating f_1 and f_2 respectively, to within the value of ϵ. Taking any pair of j, k, and assuming without loss of generality that $r^1_j \geq r^2_k$, it is easy to see that $\sup_{y_1,y_2,\alpha}[\alpha r^1_j 1_{A^1_j} + (1-\alpha) r^2_k 1_{A^2_k}]$ over $\alpha y_1 + (1-\alpha) y_2 = x$ will be a piece-wise linear function defined by the infinum and supremum of the sets A^1_j and A^2_k. Then one takes the maximum of these functions over all choices of i, j and finally a sequence of such simple functions converging to f_1 and f_2.

7. Assume that the function $f : \mathbf{R} \to \mathbf{R}$ is measurable. On the set where the second derivative f'' is well defined show that it is Lebesgue measurable.

For every i consider the function $g_i(x) = i^2(f(x + \frac{1}{i}) + f(x - \frac{1}{i}) - 2f(x))$. As each such function g_i is measurable, the function $g(x) = \lim_{i\to\infty} g_i(x)$, where it exists, is measurable. Where f'' is well defined, so is $g(x)$ and they are equal.