LTCC
"MORSE THEORY, TOPOLOGY AND ROBOTICS"
EXAM 2022 - 2023

EXAMINER: PROFESSOR MICHAEL FARBER

1.

(a) Give the definition of a vector field on a manifold,

A vector field is a function which associates a tangent vector to every point of the manifold. In other words, it is a section of the tangent bundle.

(b) What is meant by a 1-parameter group of diffeomorphisms,

A 1-parameter group of diffeomorphisms is a family $\phi_t : M \to M$ of diffeomorphisms, where $t \in \mathbb{R}$, such that $\phi_t \circ \phi_s = \phi_{t+s}$ for $t, s \in \mathbf{R}$.

(c) State the theorem about 1-parameter group of diffeomorphisms generated by a vector field.

Every smooth vector field with a compact support generates a 1-parameter group of diffeomorphisms.

2.

(a) Give the definition of a Morse critical point;

A critical point $p \in M$ of a smooth function $f : M \to \mathbf{R}$ is a Morse critical point if the Hessian matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$$

is non-degenerate.

(b) State the Morse Lemma;

Morse Lemma states that in the neighbourhood of a Morse critical point one may find a local coordinate system such that the function has the form

$$-x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

Here λ is the Morse index of p.

(c) State the theorem about changes in the sub-level set when crossing a non-degenerate critical level.

If the interval $[a, b]$ contains a single critical level $f(p)$ where p is non-degenerate in the sense of Morse and has Morse index λ then the sublevel set $M^b = f^{-1}((-\infty, b])$ is homotopy equivalent to M^a with a cell of dimension λ attached.

(d) State the Morse inequalities.

In the simplest form the Morse inequalities state that a Morse function on a compact manifold M has at least $b_i(M)$ critical points of index i, for every $i = 0, 1, \ldots, n = \dim M$.

(e) What is the minimal number of critical points of a Morse function on a closed orientable surface of genus g?

$2g + 2$.

3.

(a) Describe a Morse function on $\mathbb{CP}^n$ and find all its critical points.

Points of the complex projective space $\mathbb{CP}^n$ are equivalence classes of $(n+1)$-tuples $(z_0, z_1, \ldots, z_n) \neq 0$ where $(z_0, z_1, \ldots, z_n) \sim (z'_0, z'_1, \ldots, z'_n)$ if for some $\lambda \in \mathbb{C}^*$ one has $z_i = z'_i$. Equivalently, we may consider only the tuples $(z_0, z_1, \ldots, z_n)$ satisfying

$$\sum_{i=0}^{n} |z_i|^2 = 1$$

and define the equivalence relation $(z_0, z_1, \ldots, z_n) \sim (z'_0, z'_1, \ldots, z'_n)$ as $z_i = \lambda z'_i$ for some $\lambda = e^{i\phi} \in \mathbb{C}$ and for all $i = 0, \ldots, n$. Using the second presentation we may define a function

$$f : \mathbb{CP}^n \to \mathbb{R}$$

by $f(z_0, \ldots, z_n) = \sum_{i=0}^{n} c_i |z_i|^2$ where $c_0 < c_1 < \cdots < c_n$ is a sequence of real numbers.

In the chart $U_k \subset \mathbb{CP}^n$ defined by $z_k \neq 0$ the coordinates are given by the functions z_i with $i \neq k$ and the function in these coordinates has the form

$$f = c_k + \sum_{j \neq k} (c_j - c_k) |z_j|^2$$

We see that the point $(z_0, z_1, \ldots, z_n) \in \mathbb{CP}^n$, where $z_k = 1$ and $z_i = 0$ for all $i \neq k$, is the only critical point in U_k. Its Morse index equals $2k$.

(b) Give the definition of Lusternik - Schnirelmann category.

The number $\mathsf{cat}(X)$ is defined as the minimal integer k such that X admits an open cover $X = U_0 \cup U_1 \cup \cdots \cup U_k$ with the property that each inclusion $U_i \to X$ is null-homotopic.

(c) State the category of the sphere S^2, torus T^2, surface of genus 2?

$\mathsf{cat}(S^2) = 1$,
$\mathsf{cat}(T^2) = 2$,
$\mathsf{cat}(\Sigma) = 2$, where Σ is a compact orientable surface of genus 2.

4.

(a) Give the definition of a linkage;

A planar linkage is a mechanism consisting of several bars connected by revolving joints. The cyclic linkage consists of n bars forming a closed polygonal chain.

(b) Describe the configuration space of a planar linkage.

The configuration space of a cyclic linkage with n bars of length $\ell_1, \ell_2, \ldots, \ell_n$ is defined as

$$\{(u_1, \ldots, u_n) \in S^1 \times S^1 \times \cdots \times S^1; \sum_{i=1}^{n} \ell_i u_i = 0\}/\mathrm{SO}(2).$$

(c) What is meant by the length vector of a linkage?

The length vector of a linkage is the vector $\ell = (\ell_1, \ldots, \ell_n)$; its components are the lengths of the bars of the linkage.

(d) When do we say that the length vector is generic?

The length vector $\ell = (\ell_1, \ldots, \ell_n)$ is generic if for any choice of the numbers $\epsilon_i = \pm 1$, where $i = 1, 2, \ldots n$, one has

$$\sum_{i=1}^{n} \epsilon_i \ell_i \neq 0.$$

(e) State the theorem about Betti numbers of moduli spaces of planar linkages.

The i-dimensional Betti number of the configuration space of the linkage equals

$$a_i + a_{n-3-i} + b_i$$

where a_i is the number of short subsets of $\{1, 2, \ldots, n\}$ of cardinality $i+1$ containing 1 and b_i is the number of median subsets of $\{1, 2, \ldots, n\}$ of cardinality $i+1$ containing 1. Here we assume that $\ell_1 \geq \ell_j$ for all j.