Practice exam and solutions

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LTCC module on pseudodifferential operators and applications to PDEs

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1) (i) Let $a \in S^{m_1}(\mathbb{R}^{2n})$ and $b \in S^{m_2}(\mathbb{R}^{2n})$. Let $A = a(x, D)$ and $B = b(x, D)$. What is the order of the symbol of the commutator $AB - BA$?

   (ii) Let $a \in S^m(\mathbb{R}^{2n})$ be real-valued. Show that $AA^* - A^*A$, where $A = a(x, D)$, is a pseudodifferential operator with symbol of order $2m - 2$.

   (iii) Let $a \in S^m(\mathbb{R}^{2n})$. Assume that $A = a(x, D)$ is self-adjoint, i.e., $A = A^*$. Show that $\operatorname{Re} a$ is a symbol of order m and that $\operatorname{Im} a$ is a symbol of order $m - 1$.

2) In the sequel $x, \xi \in \mathbb{R}$.

   (i) Let
$$a(x, \xi) = (\cos x + 2)\langle \xi \rangle^m + b(x, \xi),$$
   where $b \in S^{m'}(\mathbb{R}^2)$ with $m' < m$. Prove that a is an elliptic symbol.

   (ii) Let $p(x, \xi) = \langle \xi \rangle$ and $q(x, \xi) = \sin x$. Prove that they are symbols and determine $p\sharp q$ modulo S^{-1}.

3) Let $a \in S^m(\mathbb{R}^{2n})$. Let $p, q \in S^{-m}(\mathbb{R}^{2n})$ such that the following equalities
$$p(x, D)a(x, D) = I + r(x, D), \qquad a(x, D)q(x, D) = I + s(x, D)$$
   holds on $\mathscr{S}'(\mathbb{R}^n)$ for $r, s \in S^{-\infty}(\mathbb{R}^{2n})$. Show that $p - q \in S^{-\infty}(\mathbb{R}^{2n})$.

4) Let $s \in \mathbb{R}$ and $f \in H^s(\mathbb{R}^n)$. Solve the equation
$$(1 - \Delta)u = f$$
   in $\mathscr{S}'(\mathbb{R}^n)$ by finding the inverse of the operator $1 - \Delta$.

Solutions

1) Problem 5 in attachment.

2) (i) Discussed in class. The symbol a is elliptic since $(\cos x + 2)\langle \xi \rangle$ is an elliptic symbol of order m and the order of b is strictly smaller than m. (ii) $p \in S^1$ and $q \in S^0$. From the asymptotic expansion of $p\sharp q \in S^1$ we have that
$$p\sharp q = pq + \sum_{|\alpha|=1} \frac{1}{\alpha!} \partial_\xi^\alpha p D_x^\alpha q = \sin x \langle \xi \rangle - i\partial_\xi \langle \xi \rangle \cos x$$
   modulo S^{-1}.

3) Done in class.

4) Done in class. The inverse of the operator $(1 - \Delta)$ is the pseudodifferential operator with symbol $\lambda^{-2}(\xi) = \langle \xi \rangle^{-2} = (1 + |\xi|^2)^{-1}$. Hence $u = \lambda^{-2}(D)f \in H^{s+2}(\mathbb{R}^n)$.