Reps of GL_2 lecture 4

1

Let V be an irreducible representation of $\mathrm{GL}_2(F)$ which is not one-dimensional. The quantity $c = c(V)$ occurring in Casselman's theorem is called the *conductor* of V. One can show that

$$c(V) = \begin{cases} c(\chi) + c(\psi) & V = I(\chi, \psi) \\ \max\{1, 2c(\gamma)\} & V = \mathrm{St} \otimes \gamma \\ \geq 2 & V \text{ supercuspidal.} \end{cases}$$

One can compute the isomorphism class of V from the Hecke operators acting on V^{K_c} c.f. Loeffler–Weinstein 2012.

2 Adèle groups

2.1 Adèles and Idèles of $\mathbb{Q}$

Definition 2.1.1. Define

$$\mathbb{A}_f = \prod_{\ell \text{ prime}}' \mathbb{Q}_\ell = \{(x_\ell) \in \prod_\ell \mathbb{Q}_\ell : \ x_\ell \in \mathbb{Z}_\ell \text{ for almost all } \ell\}$$

(almost all = all but finitely many). We define $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ but we won't use this much.

Topology on $\mathbb{A}_f$: open sets are products $\prod_\ell U_\ell$ of open sets such that $U_\ell = \mathbb{Z}_\ell$ for almost all ℓ. In particular

$$\hat{\mathbb{Z}} := \{(x_\ell) : x_\ell \in \mathbb{Z}_\ell \ \forall \ell\}$$

is open in $\mathbb{A}_f$ and profinite. Thus $(\mathbb{A}_f, +)$ is a locally profinite group.

Easy fact: $\mathbb{Q}$ is dense in $\mathbb{A}_f$ (by Chinese remainder theorem).

Definition 2.1.2. The *finite idèles* are define to be $\mathbb{A}_f^\times$ with the topology given by the inclusion

$$\begin{aligned} \mathbb{A}_f^\times &\hookrightarrow \mathbb{A}_f \times \mathbb{A}_f \\ x &\mapsto (x, x^{-1}) \end{aligned}$$

(exercise: show this is really different from the subspace topology on $\mathbb{A}_f$). This makes $\mathbb{A}_f^\times$ a locally profinite group.

Fact: $\mathbb{Q}^\times$ is discrete in $\mathbb{A}_f^\times$ (as $\mathbb{Q}^\times \cap \hat{\mathbb{Z}}^\times = \pm 1$ is finite).

Proposition 2.1.3. *For any $U \subset \mathbb{A}_f^\times$ open compact, the quotient $\mathbb{A}_f^\times / \mathbb{Q}_{>0}^\times U$ is finite. If $U = \hat{\mathbb{Z}}^\times$ it is trivial.*

Proof. Suffices to prove the second statement. After some unravelling this amounts to the fact that for any collection of integers n_p, p prime, with $n_p = 0$ for almost all p, there is $x \in \mathbb{Q}_{>0}^\times$ such that $v_p(x) = n_p$ for all p (let $x = \prod p^{n_p}$). □

Remark 2.1.4. For any number field F (or even any global field) one can analogously define $\mathbb{A}_F = \prod_v' F_v$ where the restricted product is with respect to $\hat{\mathcal{O}}_F = \varprojlim \mathcal{O}_F / I$.

2.2 GL_2 and SL_2

Proposition 2.2.1. *For all $N \geq 1$, the reduction map*

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N)$$

is surjective.

Proof. Everyone knows how to do this. □

Corollary 2.2.2 (Strong approximation)**.** $\mathrm{SL}_2(\mathbb{Q})$ *is dense in* $\mathrm{SL}_2(\mathbb{A}_f)$.

Proof. Closure of $\mathrm{SL}_2(\mathbb{Q})$ contains $\mathrm{SL}_2(\hat{\mathbb{Z}})$ by proposition. But Cartan decomposition for $\mathrm{SL}_2(\mathbb{Q}_N)$ implies that

$$\mathrm{SL}_2(\mathbb{A}_f) = \sqcup_{m \geq 1} \mathrm{SL}_2(\hat{\mathbb{Z}}) \left(\begin{smallmatrix} m & \\ & m^{-1} \end{smallmatrix} \right) \mathrm{SL}_2(\hat{\mathbb{Z}})$$

so the closure of $\mathrm{SL}_2(\mathbb{Q})$ is everything. □

Analogue for GL_2 fails.

2.3 Modular curves

Let $\mathcal{H}$ denote the complex upper half-plane equipped with its usual action of $\mathrm{GL}_2(\mathbb{R})^+$ by Möbius transformations. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup ($\iff$ closure $U = \bar{\Gamma}$ in $\mathrm{SL}_2(\hat{\mathbb{Z}})$ is open, and $\Gamma = U \cap \mathrm{SL}_2(\mathbb{Q})$).

Proposition 2.3.1. *Let U be as above. The double coset space*

$$\mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A}_f) \times \mathcal{H} / U$$

where the action of $\mathrm{SL}_2(\mathbb{Q})$ *is given by the diagonal left action and the action of U is given by the right action on* $\mathrm{SL}_2(\mathbb{A}_f)$*, is canonically isomorphic to $\Gamma \backslash \mathcal{H}$, via $\tau \mapsto (1, \tau)$.*

Proof. Given any $(g, \tau) \in \mathrm{SL}_2(\mathbb{A}_f) \times \mathcal{H}$, density of $\mathrm{SL}_2(\mathbb{Q})$ tells us we have $Ug^{-1} \cap \mathrm{SL}_2(\mathbb{Q}) \neq \emptyset$ and thus there is $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ such that $\gamma g \in U$. Thus

$$(g, \tau) \sim (\gamma g, \gamma \tau) \sim (1, \gamma \tau).$$

On the other hand, if $(1, \tau) \sim (1, \tau')$, then there exists $\gamma \in \mathrm{SL}_2(\mathbb{Q}), u \in U$ such that

$$(1, \tau') = (\gamma u, \gamma \tau)$$

this implies that $\gamma \in U \cap \mathrm{SL}_2(\mathbb{Q}) = \Gamma$. □

We now consider GL_2. Strong approximation fails in this case, so adèlic objects give something new. Let $U \subset \mathrm{GL}_2(\mathbb{A}_f)$ open compact, $\Gamma = I \cap \mathrm{GL}_2(\mathbb{Q})^+$ doesn't uniquely determine U. for example, consider

$$\begin{aligned} U = & \{ \left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix} \right) \bmod N \} \\ & \{ \left(\begin{smallmatrix} 1 & * \\ & * \end{smallmatrix} \right) \bmod N \} \\ & \{ \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix} \right) \bmod N \} \end{aligned}$$

all satisfy $U \cap \mathrm{GL}_2^+(\mathbb{Q}) = \Gamma_1(N)$.

Theorem 2.3.2. *Let $U \subset \mathrm{GL}_2(\mathbb{A}_f)$ open-compact. Then*

$$Y(U) := \mathrm{GL}_2^+(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathcal{H}/U$$

is a manifold with finitely many connected components, each non-canonically isomorphic to a quotient of $\mathcal{H}$.

More precisely: *Let $g_1, \ldots, g_n$ be any set of elements of $\mathrm{GL}_2(\mathbb{A}_f)$ whose determinants are representatives of $\mathbb{A}_f^\times/\mathbb{Q}_{>0}^\times \det(U)$. Then $\Gamma_i := \mathrm{GL}_2^+(\mathbb{Q}) \cap g_i U g_i^{-1}$ is a subgroup of $\mathrm{SL}_2(\mathbb{Q})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$, and*

$$\sqcup_{i=1}^n \Gamma_i \backslash \mathcal{H} \to Y(U)$$
$$\tau \text{ in } i\text{th component} \mapsto (g_i, \tau).$$

Remark 2.3.3. We can also write

$$Y(U) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \times (\mathbb{C} - \mathbb{R})/U$$

The delicate step in the proof of the above theorem is to prove that the map

$$\sqcup_{i=1}^n \Gamma_i \backslash \mathcal{H} \to Y(U)$$

is surjective i.e. that

$$\mathrm{GL}_2(\mathbb{A}_f) = \cup_{i=1}^r \mathrm{GL}_2^+(\mathbb{Q}) g_i U.$$

this comes from strong approximation for SL_2. In particular: if $\det(U) = \hat{\mathbb{Z}}^\times$, then

$$Y(U) = \Gamma\backslash\mathcal{H}, \Gamma = U \cap \mathrm{GL}_2^+(\mathbb{Q})$$

but there are many U with the same Γ in general if $U = \{\left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right) \bmod N\}, U = \{\left(\begin{smallmatrix} 1 & * \\ & * \end{smallmatrix}\right) \bmod N\}$ then $Y(U), Y(U')$ are both $\Gamma_1(N)\backslash\mathcal{H}$ but action of $\{\left(\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right) : x \in \hat{\mathbb{Z}}^\times\}$ is *different*.

3 Modular forms via adèles

3.1 Recap of modular forms

For $f : \mathcal{H} \to \mathbb{C}, g \in \mathrm{GL}_2^+(\mathbb{R}), k \in \mathbb{Z}, t \in \mathbb{R}$, define

$$\left(f|_{(k,t)} g\right) = \det(g)^t (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Definition 3.1.1. For $\Gamma \subset \mathrm{GL}_2^+(\mathbb{Q})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$ (i.e. $\Gamma \cap \mathrm{SL}_2(\mathbb{Z})$ has finite index in Γ and $\mathrm{SL}_2(\mathbb{Z})$) a modular form of level Γ and weight (k, t) is a function $f : \mathcal{H} \to \mathbb{C}$ such that:

- f is holomorphic
- $F|_{k,t}\gamma = f$ for all $\gamma \in \Gamma$.
- $(f|_{k,t}\gamma)(\tau)$ bounded as $\mathrm{Im}\tau \to \infty$ for all $\Gamma \in \mathrm{GL}_2^+(\mathbb{Q})$.

If $(f|_{k,t}\gamma)(\tau) \to 0$ for all γ we say that f is a *cusp form*. Standard fact: $S_{k,t}(\Gamma), M_{k,t}(\Gamma)$ are finite dimensional and there is a natural inner product on $S_{k,t}(\Gamma)$.

3.2 Adèlic picture

Choose $(k,t) \in \mathbb{Z} \times \mathbb{R}$ as before.

Definition 3.2.1. An adèlic modular form of weight (k,t) is a function

$$F : \mathrm{GL}_2(\mathbb{A}_f) \times \mathcal{H} \to \mathbb{C}$$

such that

1. $F(g,\tau)$ is holomorphic in τ for all g.
2. $F(gu,\tau) = F(g,\tau)$ for all $g \in \mathrm{GL}_2(\mathbb{A}_f), \tau \in \mathcal{H}, u \in U$ for some open-compact U (depending on F).
3. $F(\gamma g, -) = F(g,-)|_{k,t}\gamma^{-1}$ for all $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$.
4. For all $g \in \mathrm{GL}_2(\mathbb{A}_f)$, $F(g,\tau)$ bounded as $\mathrm{Im}(\tau) \to \infty$.

If $F(g,\tau) \to 0$ for all g say F is a cusp form.

This gives spaces $M_{k,t} \supset S_{k,t}$ which are $\mathrm{GL}_2(\mathbb{A}_f)$-representations.

Fact: These are admissible smooth and if $t = k/2$, $S_{k,t}$ is *unitarisable* i.e. there exists a G-invariant, conjugate symmetric, positive definite pairing on $S_{k,k/2}$.

Proposition 3.2.2. *If $U \subset \mathrm{GL}_2(\mathbb{A}_f)$ open compact, $g_1, \ldots, g_n$ as before, then*

$$(S_{k,t})^U = \bigoplus_{i=1}^{n} S_{k,t}(\Gamma_i)$$

via evaluation at $g_1, \ldots, g_r$ (in particular, it's finite dimensional which implies admissibility). Similary for $M_{k,t}$.

In particular, if $U = \{\left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right) \bmod N\}$ we *recover* $S_{k,t}(\Gamma_1(N))$ as invariants in an admissible smooth representation of $\mathrm{GL}_2(\mathbb{A}_f)$.

Proposition 3.2.3. *If $U_1(N)$ is above subgroup and ϖ = idèle which is 1 at all places except p and a uniformiser at p, then $[U_1(N)\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)U_1(N)]$ acts on $(M_{k,t})^{U_1(N)}$ as the classical Hecke operator $p^{1-t}U_p$ resp $p^{1-t}T_p$:*

$$[U\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)U] = p^{1-t}\begin{cases} U_p & p \mid N \\ T_p & p \nmid N \end{cases}$$

Proof. Assume $p \mid N$ first. Then

$$[U\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)U]f = \sum_{a \in \mathbb{Z}/p} \left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right) f.$$

Evaluate at $(1,\tau)$:

$$\begin{aligned} \sum_a f(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right), \tau) &= \sum_a f(\left(\begin{smallmatrix} p & a \\ & 1 \end{smallmatrix}\right), \tau) \\ &= \sum_a \left(f(1,-)|_{k,t}\left(\begin{smallmatrix} p & a \\ & 1 \end{smallmatrix}\right)^{-1}\right)(\tau) \\ &= \sum_a p^{-t} f(1, \frac{\tau - a}{p}) \\ &= p^{1-t}U_p(f(1,-))(\tau). \end{aligned}$$

Similar when $p \nmid N$ with one extra coset $\left(\begin{smallmatrix} 1 & \\ & \varpi_p \end{smallmatrix}\right)$; massage by multiplying on the right by U to get something in $\mathrm{GL}_2^+(\mathbb{Q})$. □

Similary $[U_1(N)\left(\begin{smallmatrix} \varpi & \\ & \varpi \end{smallmatrix}\right)U_1(N)], p \nmid N$ is $p^{k-2t}\langle p \rangle$ (Exercise: what if $p \mid N$??), so nebentype character of a classical modular form encodes the action of the centre.

4 Multiplicty one

4.1 Restricted tensor products

Recall 'almost all' = 'all but finitely many'. Let F be a number field.

Definition 4.1.1. Suppose we have a collection of vector spaces X_v for v a prime of F and vectors $x_v^\circ \in X_v$ (non-trivial for most v). Define $\bigotimes'_v (X_v, x_v^\circ)$ as a subspace of $\bigotimes_v X_v$ spanned by tensors $\otimes_v x_v$ such that $x_v = x_v^\circ$ for almost all v. We will often drop x_v° from the notation and write

$$\bigotimes_v{}' X_v$$

There are two key examples of this contruction:

- *Hecke algebras*: $X_v = \mathcal{H}(\mathrm{GL}_2(F_v))$, $x_v^\circ = e_{K_v}, K_v = \mathrm{GL}_2(\mathcal{O}_v)$. Then

$$\bigotimes_v{}' (X_v, x_v^\circ) = \mathcal{H}(\mathrm{GL}_2(\mathbb{A}_{F,f}))$$

- *Irreducible representations*: Let Π be an irreducible smooth $\mathrm{GL}_2(\mathbb{A}_{F,f})$-representation.

  Theorem 4.1.2 (Flath's tensor product theorem)**.** *There exist uniquely determined irreducible representations $\Pi_v \circlearrowright \mathrm{GL}_2(F_v)$ and $\phi_v^\circ \in (\pi_v)^{K_v}$ (almost all non-zero) such that*

$$\Pi \cong \bigotimes_v{}' (\Pi_v, \phi_v^\circ)\,.$$

  In particular, if $\Pi \subset S_{k,t}$ is an irreducible subrepresentation, get smooth irreducible representations Π_v for every prime v and almost all Π_v are spherical. If U is a subgroup of the form $U_v \times U^v$ with $U_v \subset \mathrm{GL}_2(F_v)$ and $U^v \subset \prod'_{w \neq v} \mathrm{GL}_2(F_w)$ then

$$\Pi^U \cong \text{Sum of finitely many copies of } \Pi_v^{U_v} \text{ as } \mathcal{H}(\mathrm{GL}_2(F_v), U_v) \text{ representation}$$

  Note that $\mathcal{H}(\mathrm{GL}_2(F_v), U_v) \subset \mathcal{H}(\mathrm{GL}_2(\mathbb{A}_{F,f}), U)$