Reps of GL_2 lecture 5

1 Multiplicity one continued

1.1 Global Kirillov models

$S_{k,t}$ as before and let $F = \mathbb{Q}$.

Definition 1.1.1. For $f \in S_{k,t}$, let ϕ_f be the function on $\mathbb{A}_f^\times$ defined by

$$\phi_f(x) = \text{coefficient of } e^{2\pi i\tau} \text{ in Fourier expansion of } f(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right), \tau)$$

where we note that the Fourier expansion can involve $e^{2\pi a\tau}$ for $a \notin \mathbb{Z}$.

Proposition 1.1.2. *1. ϕ_f supported in $\mathbb{A}_f^\times \cap$ (compact set in $\mathbb{A}_f$).*

2. For $n \in \mathbb{Q}_{>0}^\times$

$$\begin{aligned}\phi_f(nx) &= n^{-t} a_n\left(f(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right), \tau)\right) \\ &= n^{-t}(\textit{coefficient of } e^{2\pi i n\tau} \textit{ in } f(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right), \tau))\end{aligned}$$

3. $\phi_{\left(\begin{smallmatrix} a & b \\ & 1 \end{smallmatrix}\right)f}(x) = \theta(bx)\phi_f(ax)$ where $\theta : \mathbb{A}_f \to \mathbb{C}^\times$ is the unique smooth character such that $\theta(x) = e^{-2\pi i x}, x \in \mathbb{Q}$ (note that $\mathbb{A}_f/\hat{\mathbb{Z}} = \mathbb{Q}/\mathbb{Z}$).

4. $f \mapsto \phi_f$ is injective.

Proof. 1. Very similar to local version

2. Look at how $\left(\begin{smallmatrix} n & \\ & 1 \end{smallmatrix}\right)$ acts.

3. Formal if $bx \in \mathbb{Q}$ and follows for all b via strong approximation (smooth characters are locally constant).

4. Clear from 2. that ϕ_f determines $f(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right), \tau)$ for all x, τ. But $\begin{pmatrix} \mathbb{A}_f^\times & \\ & 1 \end{pmatrix}$ contains a set of representatives for $\mathrm{GL}_2^+(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f)/U$ for any open U.

□

Now let Π be an irreducible subrepresentation of $S_{k,t}$ (a *cupsidal automorphic representation* of $\mathrm{GL}_2(\mathbb{A}_f)$) of weight k, t.

Proposition 1.1.3. *1. Have $\Pi = \bigotimes'_\ell \Pi_\ell$, where Π_ℓ is infinite dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$.*

2. Consider the space

$$K(\Pi) = \{\phi_f : f \in \Pi\}$$

then $K(\Pi) \cong \bigotimes'_\ell K(\Pi_\ell, \theta_\ell)$ via the map sending $\phi_{\otimes x_\ell} \mapsto \bigotimes_\ell \phi_{x_\ell}$.

Corollary 1.1.4. *Let Π_1, Π_2 be two cuspidal automorphic representations of weight (k,t) such that*

$$\Pi_1 \cong \Pi_2$$

as $\mathrm{GL}_2(\mathbb{A}_f)$-representations. Then $\Pi_1 = \Pi_2$.

Proof. Uniqueness of Kirillov model means that if $\Pi_{1,\ell} \cong \Pi_{2,\ell}$ then $K(\Pi_{1,\ell}, \theta_\ell)$ and $K(\Pi_{2,\ell}, \theta_\ell)$ are the same function spaces on $\mathbb{Q}_\ell^\times$. If this holds for all ℓ then $K(\Pi_1) = K(\Pi_2)$. Since we can recover the function f from ϕ_f we get that $\Pi_1 = \Pi_2$ as function spaces inside $S_{k,t}$. □

Theorem 1.1.5. *(Strong multiplicity one) Let Π_1, Π_2 be cuspidal automorphic representations of same weight as before. Suppose $\Pi_{1,\ell} \cong \Pi_{2,\ell}$ for almost all ℓ. Then $\Pi_1 = \Pi_2$.*

Proof. Choose a finite set of primes S containing all ℓ such that $\Pi_{1,\ell} \neq \Pi_{2,\ell}$. For $\ell \in S$ let $\phi_{\ell,1} = \mathbb{1}_{\mathbb{Z}_\ell^\times} \in K(\Pi_{1,\ell}, \theta_\ell)$, which we can do since $C_c^\infty(G_\ell) \subset K(\Pi_{1,\ell}, \theta_\ell)$. For $\ell \notin S$ choose any $\phi_\ell \in K(\Pi_{1,\ell}) = K(\Pi_{2,\ell})$ with almost all equal to the spherical vector. Then $\phi = \otimes_\ell \phi_\ell \in K(\Pi_1) \cap K(\Pi_2)$. Since Π_1 and Π_2 are irreducible and

$$S_{k,t} \hookrightarrow (\text{functions on } \mathbb{A}_f^\times)$$

this shows $\Pi_1 \cap \Pi_2 \neq 0 \implies \Pi_1 = \Pi_2$. □

Remark 1.1.6.

- Ramakrishnan has shown that if $\Pi_1 \neq \Pi_2$ then $\{\ell : \Pi_{1,\ell} \cong \Pi_{2,\ell}\}$ has density $\leq 7/8$.
- The analogue of multiplicity one for SL_2 is true but the naïve analogue of strong multiplicity one is false.

1.2 Concrete consequences of multiplicity one

Proposition 1.2.1.

$$S_{k,t} = \bigoplus_{\substack{\Pi \ CAR \\ weight\ (k,t)}} \Pi$$

Proof. By twisting can assume $t = k/2$. Then

$$S_{k,t} \text{ unitarisable} \implies \text{direct sum of irreducibles}$$

and by multiplicity one each summand appears only once. □

This also shows summands are *orthogonal*. For each Π, let $c(\Pi) = \prod_\ell \ell^{c(\pi_\ell)}$. By Casselman's theorem

$$\dim_{\mathbb{C}} \Pi^{U_1(N)} = \begin{cases} 0 & \text{if } c(\Pi) \nmid N \\ \#\text{ of divisors of } N/c(\Pi) & \text{otherwise.} \end{cases}$$

In particular, for $N = c(\Pi)$, $\Pi^{U_1(N)}$ is one-dimensional.

Proposition 1.2.2. *This gives a bijection*

$$(\mathit{CARs\ of\ weight}\ (k,t)) \cong (\mathit{Normalised\ newforms\ in}\ S_{k,t}(\Gamma_1(N))\ \mathit{for\ some}\ N)$$
$$\Pi \mapsto \mathit{modular\ form\ spanning}\ \Pi^{U_1(N)},\ N = c(\Pi).$$

Proof. Given a CAR Π, let f_Π be any generator of $\Pi^{U_1(N)}$. Then $f_\Pi(1, -)$ is a modular form of level $N = c(\Pi)$ and is an eigenvector for T_ℓ, U_ℓ operators because these are double cosets $\left(\begin{smallmatrix} \varpi_\ell & \\ & 1 \end{smallmatrix}\right)$ up to scaling. Without loss of generality we can assume $a_1(f_\Pi) = 1$ and f_Π is new because Π is orthogonal to all CARs of conductor $< c(\Pi)$. □

Upshot: $S_k(\Gamma_1(N))$ decomposes as

$$\bigoplus_{\Pi \text{ CAR of conductor } c(\Pi) \nmid N} S_k(\Gamma_1(N))[\Pi]$$

with each summand spanned by $f(d\tau)$ for $d \mid N/c(\Pi)$ and if f_1, f_2 are two normlaised newforms such that $a_\ell(f_1) \neq a_\ell(f_2)$ for infinitely many prime ℓ. Note that if f_π is the newvector of $\Pi = \bigotimes'_\ell \Pi_\ell$, then for all $\ell \nmid$ (level of f_Π) we have $\Pi_\ell \cong I(\alpha_\ell, \beta_\ell)$ with α_ℓ, β_ℓ unramified characters sending ℓ to roots of

$$X^2 - \frac{a_\ell(f_\Pi}{\ell^{t-1/2}} + \ell^{k-2t}\varepsilon(\ell)$$

where ε is the character of f_Π.

1.3 Twisting

Recall that a Dirichlet character is a homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times.$$

We can inflate a Dirichlet character to a smooth character of $\mathbb{A}_f^\times/\mathbb{Q}^\times$ be pulling back along the natural quotient map $\hat{\mathbb{Z}}^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$. If χ is such a character, Π an automorphic representation, then can form $\Pi \otimes (\chi \circ \det)$ which we usually just refer to as $\Pi \otimes \chi$

Proposition 1.3.1. *$\Pi \otimes \chi$ is automorphic of same weight as Π.*

Proof. Let $f \in \Pi$. Consider

$$(g, \tau) \mapsto \chi(\det(g))f(g, \tau).$$

This is clearly in $S_{k,t}$ and generates a representation isomorphic to $\Pi \otimes \chi$. □

¡BUT! this function evaluated at $(1, \tau)$ is just f! So both give the same element of $S_k(\Gamma_1(N))$. What's going on here? $f \in S_k(\Gamma_1(N))$ extends uniquely to $f \in S_{k,t}^{U_1(N)}$ but there are lots of other subgroups of $\mathrm{GL}_2(\mathbb{A}_f)$ extending $\Gamma_1(N)$!

Proposition 1.3.2. *If Π CAR then the space of functions on $\mathcal{H}$, $\{f(1,\tau) \mid f \in \Pi\}$ is spanned by all the $\mathrm{GL}_2^+(\mathbb{Q})$ translates of $f_{\Pi\otimes\chi}$ for χ a Dirichlet character.*

Proof. Without loss of generality $f \in \Pi$ invariant under $M = \{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \bmod N\}$ for some N. Decompose Π^M as a representation of $\left(\begin{smallmatrix} \hat{\mathbb{Z}}^\times & \\ & 1 \end{smallmatrix}\right)$ then χ-eigenspace gives functions whose restrictions are $\mathrm{GL}_2^+(\mathbb{Q})$ translates of $f_{\Pi\otimes\chi}$. □

2 Eisenstein series

Let $\mathcal{E}_{k,t}$ be the orthogonal complement of $S_{k,t}$ in $M_{k,t}$.

2.1 Reminders

$\Gamma \subset \mathrm{SL}_2(\mathbb{Q})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1.1. The *cusps* of Γ are the finite set

$$C(\Gamma) = \Gamma\text{-orbits on } \mathbb{P}^1(\mathbb{Q}).$$

We say a cusp $c \in C(\Gamma)$ is *irregular* if $\mathrm{Stab}_\Gamma(c)$ contains an element conjugate to $\left(\begin{smallmatrix} -1 & * \\ & -1 \end{smallmatrix}\right)$ for some $*$. Otherwise all elements of the stabiliser are conjugate to $\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$ and the cusp is called *regular.* If $-1 \in \Gamma$ then all cusps are irregular!.

Let

$$\mathcal{E}_k(\Gamma) = \{f \in M_k(\Gamma) \mid \langle f, g\rangle = 0 \text{ for all } g \in S_k(\Gamma)\}.$$

Any $f \in \mathcal{E}_k(\Gamma)$ has values $f(c) \in \mathbb{C}$ for $c \in C(\Gamma)$ which gives an injection

$$\mathcal{E}_k(\Gamma) \hookrightarrow \mathbb{C} \cdot C(\Gamma)$$

sending $f \mapsto (f(c))_{c \in C(\Gamma)}$.

Fact: If k is odd, $f(c) = 0$ for all irregular cusps, so

$$\mathcal{E}_k(\Gamma) \hookrightarrow \mathbb{C}_{\text{reg}}(\Gamma)$$

Proposition 2.1.2. *1. If $k \geq 3$, then the map*

$$\mathcal{E}_k(\Gamma) \to \begin{cases} \mathbb{C} \cdot C(\Gamma) & \text{if } k \text{ even} \\ \mathbb{C} \cdot C_{\text{reg}}(\Gamma) & \text{if } k \text{ odd} \end{cases}$$

is a bijection.

2. If $k = 2$, cokernel is one-dimenisonal with unique relation being

$$\sum_c \text{width(c)} \cdot f(c) = 0$$

for all $f \in \mathcal{E}_2(\Gamma)$.

Proof. Explicit series computations. □

2.2 $\mathcal{E}_{k,t}$ as a representation of GL_2

Definition 2.2.1. For $f \in \mathcal{E}_{k,t}$ let

$$\alpha(f) = f(1, \infty)$$

where we are using that f is a modular form and therefore has a well-defined value at the cusp at infinity.

Proposition 2.2.2. *1. If $a, d \in \mathbb{Q}^\times, ad > 0$, then*

$$\alpha(\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) f) = d^k (ad)^{-t} \alpha(f)$$

2. If $x \in \mathbb{A}_f$, $\alpha(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) f) = \alpha(f)$

Proof. First is easy check, second is clear if $x \in \mathbb{Q}$ and rest follows from strong approximation. □

Notation: For $x \in \mathbb{A}_f$ let $\|x\| = \prod_\ell |x_\ell|_\ell$. Easy check, if $x \in \mathbb{Q}^\times$ then $\|x\| = |\frac{1}{x}|$.

For χ_1, χ_2 finite order characters of $\mathbb{A}_f^\times / \mathbb{Q}_{>0}^\times$, define

$$\alpha_{\chi_1,\chi_2}(f) = \int_{\left(\mathbb{A}_f^\times / \mathbb{Q}_{>0}^\times\right)^2} \chi_1(a)^{-1} \chi_2(d) \, \|d/a\|^{k/2} \, \alpha(\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) f) d(a) d(d)$$

(here we take $t = k/2$). Then α_{χ_1,χ_2} is identically zero unless

$$\chi_1(-1) = \chi_2(-1)(-1)^k.$$

α_{χ_1,χ_2} is a homomorphism of $B(\mathbb{A}_f)$-representations

$$\mathcal{E}_{k,k/2} \to \left(\chi, \|\cdot\|^{k/2}\right) \boxtimes \left(\chi_2, \|\cdot\|^{-k/2}\right)$$

or equivalently a homomorphism of $\mathrm{GL}_2(\mathbb{A}_f)$-representations

$$\begin{aligned}\mathcal{E}_{k,k/2} &\to \mathrm{Ind}_{B(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)}(\text{above character}) \\ &= \bigotimes_{\ell}{}' I\left(|\cdot|^{(k-1)/2}\chi_{1,\ell}, |\cdot|^{(1-k)/2}\chi_{2,\ell}\right)\end{aligned}$$

where the tensorands are irreducible for all ℓ if $k \neq 2$.

Proposition 2.2.3. *1. If $k \geq 3$, then the map*

$$\mathcal{E}_{k,k/2} \to \bigoplus_{(\chi_1,\chi_2):\chi_1(-1)\chi_2(-1)=(-1)^k} \bigotimes_{\ell}{}' I(\dots)$$

is an isomorphism.

2. If $k = 2$, the map is injective and its image is the kernel of the maps

$$I(\chi \|\cdot\|^{1/2}, \chi \|\cdot\|^{-1/2}) \to (\chi \circ \det)$$

for pairs of the form χ, χ

Proof. Injectivity: $f \in \mathcal{E}_{k,k/2}$ maps to 0 if and only if $\alpha(gf) = 0$. But this says $f(g, \infty) = 0$ for all g. Replace g with γg, $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ which implies $f(g, \gamma^{-1}\infty) = 0$ for all g, γ, which implies $f \in S_{k,t} \cap \mathcal{E}_{k,t} = 0$.

Surjectivity unravels to statement in classical theory about existence of Eisenstein series with specified values at the cusps. □

Upshot: If $k \geq 3$ $\mathcal{E}_{k,t}$ is a summand of distrinct, irredcible, generic (no one-dimensional local factors) representations of $\mathrm{GL}_2(\mathbb{A}_f) \implies$ oldform/newform theory works as it should. If $k = 2$ weird stuff happens: old and new subspaces not disjoint etc.

Remark 2.2.4.

- Weight 1 works but need to use unordered pairs of characters.
- Can 'put back' missing factors by using *nearly holomorphic* modular forms.