Reps of GL_2 lecture 3

1 Hecke algebras continued

As usual we let F be a non-archimedean local field and $G = \mathrm{GL}_2(F)$.

Recall that we defined the K-Hecke algebra $\mathcal{H}(G,K)$ for any open compact subgroup $K \subset G$. If $V \in \underline{Smo}_G$ is irreducible, then V^K is a simple $\mathcal{H}(G,K)$-module or 0.

Theorem 1.0.1 (Bushnell and Henniart §4.3)**.** *This gives a bijection*

$$\{\textit{Irreducible } V \textit{ such that } V^K \neq 0\}/\cong \to \{\textit{Simple } \mathcal{H}(G,K)\textit{-modules}\}/\cong$$

1.1 Spherical Hecke algebras

We take $G = \mathrm{GL}_2(F), K = \mathrm{GL}_2(\mathcal{O})$. By the Cartan decomposition the *spherical Hecke algebra* $\mathcal{H}(G,K)$ is spanned by the elements

$$[K\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}K]$$

for $a,b \in \mathbb{Z}$, so to reduce notation we'll write the above double coset as $[\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}]$. Let

$$S = \begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix}, T = \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}.$$

Theorem 1.1.1. *We have the following ring isomorphism*

$$\mathcal{H}(G,K) \cong \mathbb{C}[S, S^{-1}, T].$$

In particular, the spherical Hecke algebra is commutative.

Proof. Let $A_0 = \mathbb{C}[S,S^{-1}]$ be the central subring of $\mathcal{H}(G,K)$. We set

$$\begin{aligned} A_n &= \mathbb{C}\text{-span of } [\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}] \text{ such that } 0 \le a-b \le n \\ &= A_0\text{-span of } [\begin{pmatrix} \varpi^a & \\ & 1 \end{pmatrix}] \text{ such that } 0 \le a \le n \end{aligned}$$

Lemma 1.1.2. *For all $n \ge 1$, there is $c \ge 1$ such that*

$$T * [\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}] = c[\begin{pmatrix} \varpi^{n+1} & \\ & 1 \end{pmatrix}] \bmod A_{n-1}$$

Proof. Any double coset in the support of $T * [\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}]$ has determinant ϖ^{n+1} up to units (c.f. the definition of c_γ) and is in $M_{2\times 2}(\mathcal{O})$ so it is either $[\begin{pmatrix} \varpi^{n+1} & \\ & 1 \end{pmatrix}]$ or $[\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}]$ for $a+b = n+1, a \ge b \ge 1$ the former is definitely in the support, so $c \neq 0$ and the latter is in A_{n-1} for all a,b. □

We now claim that A_n is spanned as an A_0-module by $(1, T, \ldots, T^n)$ and that these are A_0-linearly independent. This is clear for $n = 0,1$ and follows for all n by induction using the above lemma. □

1.2 Unramified principal series

Let χ, ψ be smooth unramfied characters of $F^\times$ (i.e. restricton to $\mathcal{O}^\times$ is trivial, ergo they are of the form $\chi(x) = \alpha^{v(x)}, \psi(x) = \beta^{v(x)}$ where $\alpha = \chi(\varpi), \beta = \psi(\varpi)$).

Proposition 1.2.1. *$I(\chi, \psi)^K$ is one-dimensional and S acts as $\alpha\beta$, T as $q^{1/2}(\alpha + \beta)$.*

Proof. Since $G = BK$ by the Iwasawa decomposition, $I(-)^K$ has dimension ≤ 1 for any χ, ψ. However, we can write down an element $f_{\text{sph}} \in I(\chi, \psi)$ given by

$$f_{\text{sph}}(bk) = |\frac{a}{d}|^{1/2}\chi(a)\psi(d)$$

for $b = \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right)$, which is well defined since if $b \in K$ then $a, d \in \mathcal{O}^\times$ so $\chi(a), \psi(d)$ and $|\frac{a}{d}|^{1/2}$ are all trivial (here we use that χ, ψ are unramified). This gives an isomorphism

$$I(\chi, \psi)^K = (\chi \boxtimes \psi)^{B \cap K}$$

The action of S is clear (since S is central). We then have

$$\begin{aligned}
(T * f_{\text{sph}})(1) &= \int_G T(g)(g \cdot f_{\text{sph}})(1)dg \\
&= \frac{1}{\mu(K)} \int_{K\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)K} f_{\text{sph}}(g) \\
&= \sum_{g \in K\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)K/K} f_{\text{sph}}(g)dg \\
&= \sum_{a \in \mathcal{O}/\wp} f_{\text{sph}}(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)) + f_{\text{sph}}(\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right))
\end{aligned}$$

then $f_{\text{sph}}(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)) = q^{-1/2}\alpha f_{\text{sph}}(1)$ and $f_{\text{sph}}(\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)) = q^{1/2}\beta f_{\text{sph}}(1)$, so $(T * f_{\text{sph}})(1) = (q(q^{-1/2}f_{\text{sph}}(1)) + q^{1/2}\beta)f_{\text{sph}}(1)$ □

Equivalently, consider 'Satake polynomial'

$$X^2 - q^{-1/2}TX + S \in \mathcal{H}(G, K)[X]$$

then α, β are the roots of the Satake polynomial acting on $I(\chi, \psi)^K$.

Corollary 1.2.2. *Every irreducible representation V of G such that $V^K \neq 0$ is one of the following:*

- *$I(\chi, \psi)$ for some unramified χ, ψ satisfying $\chi/\psi \neq |\cdot|^{\pm 1}$*
- *One-dimensional representations $\chi \circ \det$ for χ unramified.*

Proof. These exhaust all possible simple $\mathcal{H}(G, K)$-modules (since $\mathcal{H}(G, K)$ is commutative all simple modules are one-dimensional). More explicitly, a simple $\mathcal{H}(G, K)$-module is given by a homomorphism $\theta : \mathcal{H}(G, K) \to \mathbb{C}$. Under the isomorphism $\mathcal{H}(G, K) \cong \mathbb{C}[S, S^{-1}, T]$ this corresponds to choosing values $\gamma = \theta(S), \delta = \theta(T)$. By considering the polynomial

$$X^2 - q^{-1/2}\delta X + \gamma$$

we see there exist complex numbers α, β such that $\gamma = \alpha\beta, \delta = q^{1/2}(\alpha + \beta)$. When $\alpha/\beta \neq q^{\pm 1}$ these correspond to spherical principal series representations. Otherwise its not hard to see that these correspond to representations of the second type. □

1.3 The Iwahori Hecke algebra

Let $I = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K : c \in \wp\}$, the *Iwahori subgroup* of G.

Proposition 1.3.1. *1. If $V = I(\chi, \psi)$ for χ, ψ unramified, $\chi/\psi \neq |\cdot|^{\pm}$, then V^I is two-dimensional.*

2. $U := [I\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)I]$ acts with eigenvalues $\{q^{1/2}\alpha, q^{1/2}\beta\}$

Proof. Assume that $\alpha \neq \beta$. I has two orbits on $\mathbb{P}^1(\mathcal{O})$ (lift the Bruhat decomposition for $\mathrm{GL}_2(k)$), represented by 1 and $w = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$ we therefore have the following decomposition

$$V^I = (\chi \boxtimes \psi\delta^{-1/2})^{B \cap I} \oplus (\chi \boxtimes \psi\delta^{-1/2})^{B \cap wIw^{-1}}$$

which is two-dimensional. We compute that if $f \in V^I$,

$$\begin{aligned} [I\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)I] * f &= \sum_{a \in \mathcal{O}/\wp} f(\left(\begin{smallmatrix} \varpi & a \\ & 1 \end{smallmatrix}\right)) \\ &= q \cdot q^{-1/2}\alpha f(1) \end{aligned}$$

so evaluation at 1 is a non-zero linear functional on V^I factoring through projection to the $U = q^{1/2}\alpha$ eigenspace. By symmetry, $q^{1/2}\beta$ eigenspace is also non-zero. If $\alpha = \beta$, U acts with matrix $q^{1/2}\left(\begin{smallmatrix} \alpha & 1 \\ & \alpha \end{smallmatrix}\right)$ (in particular, not semisimply). ☐

2 Local new vectors

2.1 Statement

Let V be an irreducible representation of G. Assume V is not one-dimensional. Let $K_n = \{g \in K : g \equiv \left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right) \bmod \wp^n\}$ (some maniacs might refer to this as a mirahoric subgroup).

Theorem 2.1.1 (Casselman)**.** *1. There is n such that $V^{K_n} \neq 0$*

2. If c is the smallest such n such that $V^{K_n} \neq 0$ then V^{K_n} is one-dimensional. Let v be any basis vector for this space (new vector).

3. For all $n \geq c$, V^{K_n} has dimension $n - c + 1$ and $\{\left(\begin{smallmatrix} 1 & \\ & \varpi^n \end{smallmatrix}\right)v : 0 \leq a \leq n - c\}$ is a basis of V^{K_n}.

2.2 The Kirillov model

As usual, let $N = \{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)\} \cong (F, +)$.

Lemma 2.2.1. *1. Let $0 \neq W \in \underline{Smo}_N$. Then there exists a character $\theta : N \to \mathbb{C}^\times$, and $\psi \in \mathrm{Hom}_N(W, \theta)$ such that $\psi \neq 0$. Moreover, for any non-zero $w \in W$ there is ψ and θ, as above such that $\psi(w) \neq 0$*

2. For V as in the previous theorem, $V^N = 0$.

Proof. 1. this reduces to showing that for all N_0 open-compact there is θ such that

$$p_\theta(w) := \int_{N_0} \theta(n)^{-1} nw dn \neq 0.$$

Let $N_1 \subset N_0$ be an open-compact subgroup fixing w. Then w generates a non-trivial finite dimensional complex representation of the finite abelian group N_0/N_1 and we observe that the map p_θ is the projection to the θ-isotypic component in this representation. As N_0/N_1 is abelian there must be a character occurring as an irreducible component in this representation.

2. See Bump Proposition 4.4.6

□

The following is often called the 'local multiplicity one' theorem.

Theorem 2.2.2 (Kirillov)**.** *For any non-trivial character θ of N*

$$\dim \mathrm{Hom}_N(V, \theta) = 1$$

Proofs are in Bump or Jacquet–Langlands.

Fix any non-trivial character θ, then every other character of N is of the form $x \mapsto \theta(ax)$ for some $a \in F$.

Definition 2.2.3. Fix a basis λ of $\mathrm{Hom}_N(V, \theta)$. For $v \in V$, let ϕ_v be the function on $F^\times$ defined by

$$\phi_v(a) = \lambda(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)v)$$

the *Kirillov function* of v. We write

$$K(V, \theta) = \{\phi_v : v \in V\}.$$

One should think of this as a 'Fourier coefficient' of V.

Proposition 2.2.4. *1. If $v \in V$, $\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \in B$, then*

$$\phi_{\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right)v}(x) = \omega_V(d)\theta(\frac{bx}{d})\phi_v(\frac{ax}{d})$$

where ω_V is the central character of V.

2. $v \mapsto \phi_v$ is an injection.

3. Any $\phi \in K(V)$ is supported on $F^\times \cap$ (compact set in F)

4. $K(V) \supset C_c^\infty(F^\times)$

5. $K(V)/C_c^\infty(F^\times)$ is finite-dimensional and is 0 if and only if V is supercuspidal.

Remark 2.2.5. 1. and 4. imply that all supercuspidal representations are isomorphic as reps of $\left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right)$ (mirabolic subgroup).

Proof. 1. Immediate from definitions

2. If $\phi_v = 0$, then $\psi(v) = 0$ for all homomorphisms ψ to non-trivial characters of N. Let $w = nv - v, n \in N$. This now dies in *every* N character quotient which implies $w = 0$ by the lemma, which implies that $v \in V^N = 0$.

3. Since V is smooth, $v = \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)v$ for some $b \neq 0$, so $\phi_v(x) = \theta(bx)\phi_v(x)$ for all x. For $v(x) \ll 0, \theta(bx) \neq 1$, so $\phi_v(x)$ must be zero.

4. We first show that $K(N)/K(N) \cap C_c^\infty$ is finite-dimensional: consider $V(N) = \langle n \cdot v - v : v \in V, n \in N\rangle \subset V$ so that $V_N = V/V(N)$ is finite dimensional (and trivial for V supercuspidal). Hence $V(N) \neq 0$. We compute that $\phi_{\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)v - v}(x) = (\theta(bx) - 1)\phi_v(x)$ and since for $v(x) \gg 0$ we have $\theta(bx) - 1 = 0$ we conclude that $\phi_v \in C_c^\infty$ for $v \in V(N)$.

5. Since $V(N) \neq 0$, $K(V) \cap C_c^\infty(F^\times) \neq 0$ but $C_c^\infty(F^\times)$ is irreducible as a B-representation.

□

2.3 Proof of Casselman's theorem

Proposition 2.3.1. *For $n \in \mathbb{Z}$ let $\bar{N}_n = \{\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right) : x \in \wp^n\}$. then $\bar{N}_n$ and $K_\infty := \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ & \mathcal{O}^\times \end{pmatrix}$ generate K_n for $n \geq 0$ and if $n \leq -1$ they generate a group containing* $\mathrm{SL}_2(F)$

Proof. Exercise. □

Now let V be as in the theorem, $v \mapsto \phi_v$ a Kirillov model and *we take the character θ to be trivial on $\mathcal{O}$ but not on $\varpi^{-1}\mathcal{O}$* because we can. It's then an easy exercise to show that:

$$v \in V^{K_\infty} \iff \phi_v \text{ supported on } \mathcal{O} \text{ and constant on cosets of } \mathcal{O}^\times$$
$$v \in V^{K_n} \iff v \in V^{K_\infty} \text{ and } v \in V^{\bar{N}_n} \text{ for } n \geq 0$$

and if $v \in V^{K_\infty} \cap V^{\bar{N}_{-1}}$, then $v = 0$.

Lemma 2.3.2 (Key Lemma)**.** $\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)$ *maps V^{K_n} to a subspace of $V^{K_{n+1}}$ of codimension ≤ 1.*

Proof. If $V^{K_{n+1}}$ and $\phi_v(1) = 0$ then ϕ_v is identically zero on $\mathcal{O}^\times$. Hence $\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)^{-1} v$ corresponds to a function on $F^\times$ supported on $\mathcal{O}$ and stable under scaling by $\mathcal{O}^\times$ and thus preserved by K_∞ and also stable under $\bar{N}_n$ therefore $\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)^{-1} v \in V^{K_n}$. So the subspace of $V^{K_{n+1}}$ such that $\phi_v(1) = 0$ is in the image of $\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)$ on V^{K_n}. □

Proof of Casselman's theorem. Let $c = \min\{n : V^{K_n} \neq 0\}$. The Key Lemma implies that V^{K_n} is one dimensional and if v is a basis then $\phi_v(1) \neq 0$ (exercise!) which implies that the codimension in the Key Lemma is 1 for all $n \geq c$ and v is a basis of $V^{K_{n+1}}/\{\text{image of } V^{K_n} \text{ by } \left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)\}$. □