Reps of GL_2 lecture 1

1 Intro

This course gives an introduction to the representation theoretic study of automorphic forms. In an introductory course in modular forms one studies modular forms as holomorphic functions on the complex upper half plane and the action of a certain algebra of operators, the Hecke algebra, on spaces of such functions. The objects and definitions occurring in such a course can seem rather ad hoc. This course will offer a first glimpse at a wider framework in which these objects naturally arise.

With a view towards further research, this course offers a first look into the language of the Langlands programme, one of the largest areas of research in modern number theory.

Disclaimer: These notes are adapted from notes from a course given by David Loeffler in Autumn 2018 at the University of Warwick. I have endeavoured to add some extra details where the original notes were, by necessity, a little terse. Any mistakes are mine.

2 Representations of locally profinite groups

2.1 Definitions

Definition 2.1.1. A topological group G is a group that is also a topological space, such that the group operation map

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

and the inversion map

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are continuous (group object in category of topological spaces).

Example 2.1.2.
- Literally any group equipped with the discrete topology
- $(\mathbb{R}, +)$, $(\mathbb{R}^\times, \times)$, $(\mathbb{Q}_p, +)$ etc with usual topology.
- $\mathbb{Q}_p$-points of algebraic groups e.g. $\mathrm{GL}_2(\mathbb{Q}_p)$, $\mathrm{GSp}_4(\mathbb{Q}_p)$ etc

FACT: in any topological group, open subgroups are closed (easy exercise)

Definition 2.1.3. G is locally profinite if every open neighbourhood of 1_G contains an open compact subgroup.

EXERCISE: If G is locally profinite, $H \leq G$ closed, then H is locally profinite and if H normal so is G/H.

Proposition 2.1.4. *If G is locally profinite, any open compact $K \subset G$ is profinite i.e. $K \mapsto \varprojlim_U K/U$ is an isomorphism of topological groups, where U runs over all open compact normal subgroups of K.*

Proof. Exercise. □

FACT: G is locally profinite if and only if G is locally compact and totally disconnected.

Example 2.1.5.

Any discrete group

Any profinite group even something dumb like $\prod_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}$.

2.2 Local fields

We fix the notation of this section.

- Let F be a non-archimedean local field with ring of integers $\mathcal{O}$, a complete DVR with max ideal $\mathfrak{m}$.
- Let v be a valuation on F normalized so that a generator ϖ of the maximal ideal of $\mathcal{O}$ has $v(\varpi) = 1$.
- $\mathcal{O}/\mathfrak{m}$ is finite of order q (a prime power).
- Absolute value $|\cdot| : F \to \mathbb{R}_{\geq 0}$ given by $|x| = q^{-v(x)}$.

Proposition 2.2.1. $\mathrm{GL}_n(F)$ *is a locally profinite topological group.*

Proof. Let

$$K_m = \{g \in \mathrm{GL}_n(\mathcal{O}) : g \equiv \ 1 \bmod q^m\},$$

then these subgroups form a basis of neighbourhoods of the identity in $\mathrm{GL}_n(F)$. These subgroups are also open compact: We first claim that $\mathrm{GL}_n(\mathcal{O})$ is open compact, which follows from taking the natural faithful representation $\mathrm{GL}_n(F) \hookrightarrow M_n(F) \cong F^{n^2}$. Then $M_n(\mathcal{O}) \cong \mathcal{O}^{n^2}$ is an open-compact subgroup of $M_n(F)$ and thus $\mathrm{GL}_n(\mathcal{O}) = M_n(\mathcal{O}) \cap \mathrm{GL}_n(F)$ is open-compact. It then suffices to show that K_m is open-compact in $\mathrm{GL}_n(\mathcal{O})$, but this follows easily by realising K_m as the kernel of reduction mod q^m which gives closedness (and thus compactness) and noting that a finite-index closed subgroup is open (same argument as open subgroup is closed). □

2.3 Smooth and admissible representations

Let G be a locally profinite group.

Definition 2.3.1. A representations of G is a $\mathbb{C}$-vector space with a left action of G by linear maps i.e. a group homomorphism

$$G \to \mathrm{Aut}(V).$$

- V is *smooth* if every $v \in V$ has open stabiliser in G.
- V is admissible if for all $K \subset G$ open compact, V^K is finite-dimensional.

Note that the topology of $\mathbb{C}$ plays no role, could take any other algebraically closed field.

Definition 2.3.2.

- $\underline{Rep}_G$ is the category of G-representations.
- $\underline{Smo}_G$ is the category of smooth G-representations
- $\underline{Adm}_G$ is the category of admissible smooth G-representations.

If $V \in \underline{Rep}_G$ then $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is in $\underline{Rep}_G$ but doesn't preserve smooth representations.

Definition 2.3.3. Let $V \in \underline{Rep}_G$ and define

$$V^\vee = \{\lambda \in V^* : \exists \text{ open } K \subset G \text{ such that } \lambda \in (V^*)^K\} \in \underline{Smo}_G$$

FACT: If $V \in \underline{Smo}_G$, natural map $V \to (V^\vee)^\vee$ exists and is injective. It's an isomorphism if and only if $V \in \underline{Adm}_G$.

Exercise: Let $G = \mathbb{Z}_p^\times$, $V = \{\text{locally constant functions } G \to \mathbb{C}\}$

- Show V is smooth and admissible
- Show that V^* is not smooth
- Show that $V^\vee$ is non-canonically isomorphic to V.

There is a version of Schur's lemma for smooth representations.

Theorem 2.3.4 (Jacquet)**.** *Suppose G/K is countable for some (therefore any) open compact subgroup $K \subset G$. If $V \in \underline{Smo}_G$ is irreducible then* $\mathrm{End}_G(V) = \mathbb{C}$.

Proof. Let $\phi \neq 0 \in \mathrm{End}_G(V)$, then $\mathrm{Ker}(\phi) = 0$ and $\mathrm{Im}(\phi) = V$ by irreducibility, so $\mathrm{End}_G(V)$ is a division algebra containing the field $\mathbb{C}(\phi)$. If $\phi \notin \mathbb{C}$ then ϕ must be transcendental over $\mathbb{C}$.

Lemma 2.3.5. *The set $\{\phi - a : a \in \mathbb{C}\}$ is linearly independent over* $\mathbb{C}$.

Proof. Any linear relation among the given set gives a polynomial which annihilates ϕ. □

But if $v \neq 0$ then by smoothness there is an open-compact subgroup $K \subset G$ such that $v \in V^K$ and the set $\{g \cdot v : g \in G/K\}$ is a countable spanning set for V so $\dim(V)$ is countable. Evaluation at v gives an injective map $\mathrm{End}_G(V) \hookrightarrow V$ which is a contradiction. □

Corollary 2.3.6. *If G/K is countable and V irreducible then the centre $Z(G)$ of G acts on V by a character* $Z(G) \to \mathbb{C}^\times$.

Proof. An element of the centre defines an element of $\mathbb{C}^\times \cong \mathrm{Aut}_G(V) \subset \mathrm{End}_G(V) \cong \mathbb{C}$. Thus for each $z \in Z(G)$ there is $\chi(z) \in \mathbb{C}^\times$ such that $z \cdot v = \chi(z)v$ for all $v \in V$ and this defines a character $\chi : Z(G) \to \mathbb{C}^\times$. □

2.4 Induced representations

Let G be a locally profinite group, $H \leq G$ a closed subgroup. There is a natural restriction functor

$$\mathrm{Res}_H^G : \underline{Smo}_G \to \underline{Smo}_H$$

can we *go the other way*?

Definition 2.4.1. For $W \in \underline{Smo}_H$, define

$$\mathrm{Ind}_H^G(W) = \{f : G \to W : f(hg) = hf(g) \ \forall \ h \in H, \text{and} \exists \text{ open-compact } K \subset G \text{ s.t. } f(gk) = f(g) \ \forall \ k \in K\},$$

with G acting by right translation.

Thus Ind_H^G is a functor $\underline{Smo}_H \to \underline{Smo}_G$.

Theorem 2.4.2 (Frobenius reciprocity)**.** *For $W \in \underline{Smo}_H, V \in \underline{Smo}_G$,*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) = \mathrm{Hom}_H(V, W).$$

i.e. induction is right adjoint to restriction.

We can also consider a variant on the above induction functor, the *compact induction* functor, defined to be

$$c - \mathrm{Ind}_H^G(W) = \text{functions in } \mathrm{Ind}_H^G(W) \text{ with compact support mod } H.$$

We will see later that this functor is left adjoint to restriction *if* H is open in G.

Proposition 2.4.3. *If G/H is compact, the functor $\mathrm{Ind}_H^G = c - \mathrm{Ind}_H^G$ sends admissible reps to admissible reps.*

Proof. Let $W \in \underline{Adm}_H$ and $V = \mathrm{Ind}_H^G(W)$. Let $K \subset G$ be an open compact subgroup. We want to show that V^K is finite-dimensional. The assumption on H means that the double quotient $H\backslash G/K$ is finite. Let $x_1, \ldots, x_n$ be a set of representatives. Then any $f \in V^K$ is determined uniquely by the set of values $f(x_i)$ for $i = 1, \ldots, n$. But

$$f(x_i) \in W^{H \cap x_i^{-1} K x_i}$$

for each i and since $H \cap x_i K x_i^{-1}$ is open-compact in H we have that the RHS is finite-dimensional by admissibility. The result follows from the injection

$$V^K \hookrightarrow \bigoplus_{i=1}^n W^{H \cap x_i K x_i^{-1}}.$$

□

An important class of group-subgroup pairs (G, H) such that G/H is compact is given (somewhat tautologically) by pairs where G is the F-points of a reductive algebraic group (e.g. $\mathrm{GL}_n, \mathrm{GSp}_4, U(2,1)$ etc) and H is the F points of a parabolic subgroup of G (e.g. for GL_n take any block upper triangular subgroup).

2.5 Haar measure and the modulus character

Theorem 2.5.1 (Haar)**.** *If H is a locally profinite group then there exists a finitely additive function*

$$\mu : \{\textit{open subsets of } G\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

finite on open-compact sets, non-zero on non-empty sets and such that

$$\mu(gS) = \mu(S)$$

for all $g \in G$. This function is called a left Haar measure and is unique up to scaling by $\mathbb{R}_{>0}^{\times}$. (For locally profinite groups choose an open compact K, give it measure 1, then $\mu(L)$ determined for every $L \subset K$ open compact subgroup).

Any left Haar measure gives a map (a *Haar integral*)

$$\int_G d\mu : C_c^\infty(G) \to \mathbb{C}$$

where $C_c^\infty(G)$ is the space of locally constant compactly supported $\mathbb{C}$-valued functions on G. We write $\int_G f(g) d\mu(g)$ for its value at $f \in C_c^\infty(G)$

Proposition 2.5.2. *If G locally profinite, there exists a character $\delta_G : G \to \mathbb{C}^\times$ such that for any left Haar measure μ,*

$$\mu(Sg) = \delta(g)\mu(S)$$

for all $g \in G$, S open.

Proof. The map $S \mapsto \mu(Sg)$ is also a left Haar measure by uniqueness there is $\delta(g) \in \mathbb{C}^\times$ such that $\delta(g)\mu(S) = \mu(Sg)$ for all S open and $g \in G$. Thus for all open S and $g, h \in G$

$$\delta(g)\delta(h)\mu(S) = \delta(g)\mu(Sh) = \mu(Sgh) = \delta(gh)\mu(S)$$

whence we conclude. □

2.6 Duality theorem

The aim of this section is to understand the dual of $c - \mathrm{Ind}_H^G(V)$. The idea is to 'integrate over $H\backslash G$' i.e. we would like a G-invariant measure on $H\backslash G$ or, equivalently, a linear functional on $C_c^\infty(H\backslash G)$, the space of locally constant compactly-supported functions on $H\backslash G$.

Proposition 2.6.1. *Let $\theta : H \to \mathbb{C}^\times$ be a smooth character. Then there exists a non-zero G-equivariant map*

$$c - \mathrm{Ind}_H^G(\theta) \to \mathbb{C}$$

if and only if $\theta = \delta_H^{-1} \cdot \delta_G|_H$. If the map exists its unique up to scalars.

Proof. See Bushnell and Henniart §3.4. Idea: The linear map

$$C_c^\infty(G) \to c - \mathrm{Ind}_H^G(\theta)$$
$$f \mapsto \tilde{f}$$

where $\tilde{f}(g) = \int_H \theta\delta_H(h)f(hg)\mu_H(h)$ is surjective and so a non-zero functional on $c - \mathrm{Ind}_H^G(\theta)$ pulls back to non-zero functional on $C_c^\infty(G)$. Since the space of functionals on $C_c^\infty(G)$ is one-dimensional and generated by the Haar integral, there is a non-zero functional on $c - \mathrm{Ind}_H^G(\theta)$ iff the Haar integral on G factors through the kernel of the above map. This occurs iff θ is of the above form. ☐

Write the functional of the proposition as

$$f \mapsto \int_{H\backslash G} f(g) d\mu_{H\backslash G}(g),$$

where $f : G \to \mathbb{C}$ transforms via $\delta_H^{-1}\delta_G$ on the left.

Theorem 2.6.2 (Duality theorem)**.** *For $V \in \underline{Smo}_H$, we have*

$$\left(c - \mathrm{Ind}_H^G(V)\right)^\vee = \mathrm{Ind}_H^G(V^\vee \otimes \delta_H^{-1}\delta_G)$$

Proof. Can define a pairing

$$c - \mathrm{Ind}_H^G(V) \times \mathrm{Ind}_H^G(V^\vee \otimes \delta_H^{-1}\delta_G) \to c - \mathrm{Ind}_H^G(\delta_H^{-1}\delta_G) \xrightarrow{\int_{H\backslash G}} \mathbb{C}.$$

This turns out to be a perfect pairing. ☐

We define the *normalized induction* of V to be

$$I_H^G(V) := \mathrm{Ind}\left((\delta_H^{-1} \cdot \delta_G)^{1/2} \otimes V\right).$$

If $H\backslash G$ is compact, we have

$$I_H^G(V)^\vee = I_H^G(V^\vee)$$