Reps of GL_2 lecture 2

1 The principal series

As usual let F be a non-archimedean local field and let $q, |\cdot|$ etc. be as before.

1.1 Some subgroups and decompositions

Let

- $B = \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$ is the *Borel subgroup*, $B = T \ltimes N$, $T = \left(\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$.
- $K_0 = \mathrm{GL}_2(\mathcal{O})$

Proposition 1.1.1. *1. $G = B \cup BwB$, where $w = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$, this is the Bruhat decomposition of G.*

2. $G = BK_0$, this is the Iwasawa decomposition

3. $G = \sqcup_{a \leq b \in \mathbb{Z}} K_0 \left(\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right) K_0$, this is the Cartan decomposition

For part 2 of the above proposition note that $G/B = \mathbb{P}^1(F) = \mathbb{P}^1(\mathcal{O}) = K_0/K_0 \cap B$.

Proposition 1.1.2. • *B is not unimodular*

- *G is unimodular*

Proof. For the first part we use the decomposition $B = T \ltimes N$. The restriction of δ_B to N has to be trivial since every element of N is contained in an open compact subgroup (indeed $N \cong F$ and so every element of N is contained in $\varpi^n N$ for some $n \in \mathbb{Z}$ and this is a subgroup of $(T \cap K_0) \times \varpi^n N$). It's also clearly trivial on the centre. Consider the element $g = \left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)$ and let $B_0 = B \cap K_0$. If $b = \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right)$ then $g^{-1}bg = \left(\begin{smallmatrix} a & \varpi b \\ & d \end{smallmatrix}\right))$. We then compute

$$\delta_B(g) = \frac{\mu(B_0 g)}{\mu(B_0)} = \frac{\mu(g^{-1}B_0 g)}{\mu(B_0)}$$

and since $[B_0 : g^{-1}B_0 g] = q$ we have $\delta_B(g) = 1/q = \#\mathcal{O}/\varpi\mathcal{O}$. More generally we have $\delta_B(\left(\begin{smallmatrix} r & s \\ & t \end{smallmatrix}\right)) = |t/r|$.

For the second part we note that, via the Cartan decomposition, it is sufficient to show that

$$\delta_G(\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)) = 1.$$

Let $U = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0 : c \in \wp\}$, then g conjugates U to $\bar{U}$, the subgroup defined instead by the condition that $b \in \wp$. Then $\bar{U}$ has the same index as U in K_0 so they have the same Haar measure whence we conclude that $\delta_G(g) = 1$. □

We remark that the F-points of reductive groups are always unimodular.

1.2 The representations $I(\chi, \psi)$

Let $\chi, \psi : F^\times \to \mathbb{C}^\times$ be smooth characters (which in practice means that $\chi|_{\mathcal{O}^\times}$ factors through $(\mathcal{O}/\varpi^n)^\times$ for some n). The value $\chi(\varpi)$ can be chosen arbitrarily. We can then define a character of the Borel B via

$$\begin{aligned} \chi \boxtimes \psi : B &\to \mathbb{C}^\times \\ \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) &\mapsto \chi(a)\psi(d). \end{aligned}$$

We set

$$I(\chi, \psi) = I_B^G(\chi \boxtimes \psi) = \{f : G \to \mathbb{C} : f(bg) = \chi(a)\psi(d)|a/d|^{1/2} f(g) \text{ and } \exists\ K \text{ open compact such that } f \in I(\chi, \psi)^K\}$$

Exercise: Show that if the first condition holds, the second condition is equivalent to the function be ing locally constant.

Proposition 1.2.1. *1. $I(\chi, \psi)$ is smooth and admissible as a G-representation.*

2. $I(\chi^{-1}, \psi^{-1}) = I(\chi, \psi)^\vee$

3. $I(\chi, \psi)$ has central character $\chi\psi$.

Proposition 1.2.2. *If $\chi/\psi = |\cdot|^{\pm 1}$, $I(\chi, \psi)$ reducible.*

Proof. If $\chi/\psi = |\cdot|^{-1}$, then there exists a 1-dimensional subspace spanned by the function

$$f(g) = |\det(g)|^{1/2}\psi(\det(g)).$$

By duality there is a 1-dimensional quotient if $\chi/\psi = |\cdot|$. □

Definition 1.2.3. The Steinberg representation St is the kernel of the map $I(|\cdot|^{1/2}, |\cdot|^{-1/2}) \to \mathbb{C}$.

Theorem 1.2.4. *1. If $\chi/\psi \neq |\cdot|^{\pm 1}$, $I(\chi, \psi)$ is irreducible and $I(\chi, \psi) \cong I(\psi, \chi)$.*

2. If $\chi/\psi = |\cdot|$, then $I(\chi, \psi)$ has a codimension 1 subrepresentation $St \otimes \psi|\cdot|^{1/2}$ and this is irreducible.

3. If $\chi/\psi = |\cdot|^{-1}$, $I(\chi, \psi)$ has a 1-dimensional subrepresentation and the quotient is $St \otimes \psi|\cdot|^{-1/2}$ (in particular $St = St^\vee$).

4. There are no further isomorphisms between these representations.

Corollary 1.2.5. *Let V be an irreducible representation of G. Then the following are equivalent:*

1. V is a subquotient of some $I(\chi, \psi)$

2. V is a subspace of some $I(\chi, \psi)$

3. The N-coinvariants $V_N \neq 0$.

Proof. 1 $\implies$ 2 is from the theorem, 2 $\implies$ 3 is from Frobenius reciprocity. □

If V does not satisfy these conditions we say that V is *supercuspidal*. Note that we have a complete classification of non-supercuspidal irreducible representations: every such representation is of the form

- $I(\chi, \psi)$ for $\chi/\psi \neq |\cdot|^{\pm 1}$
- St $\otimes\, \chi$ for a smooth character $\chi : F^\times \to \mathbb{C}^\times$
- A one-dimensional representation $\chi \circ \det$.

1.3 Jacquet modules and maps between induced representations

Lemma 1.3.1. *1. Let $V \in \underline{Smo}_N$. Then the kernel of the natural map $V \to V_N$ is given by*

$$\{v \in V : \int_{N_0} n \cdot v d\mu_N(n) = 0 \text{ for some open compact } N_0 \subset N\}$$

2. $V \mapsto V_N$ is an exact functor on $\underline{Smo}_N$ (right exactness is obvious).

Definition 1.3.2. For $V \in \underline{Smo}_B$ let $J_B(V) = V_N \otimes \delta_B^{1/2} \in \underline{Smo}_T$, the *Jacquet module* associated to V.

Then $\mathrm{Hom}_G(V, I_B^G(\chi \boxtimes \psi)) = \mathrm{Hom}_T(J_B(V), \chi \boxtimes \psi)$.

Proposition 1.3.3. *Let $\eta = \chi \boxtimes \psi$ character of T and $\eta^w = \psi \boxtimes \chi$. Then there is a short exact sequence of B-representations*

$$0 \to V \to I(\eta)|_B \to \eta \otimes \delta_B^{-1/2} \to 0.$$

Moreover $V \cong c - \mathrm{Ind}_T^B(\eta^w \otimes \delta_B^{1/2})$

Proof. Evaluation at 1_G gives a surjective B-homomorphism $I(\eta) \to \eta \otimes \delta_B^{-1/2}$. If $f \in \ker(\text{this map})$ then *support*$(f) \subset \mathbb{P}^1(F)$ has to be distinct from some open neighbourhood of 1_G, hence contained in some open compact subset of $\mathbb{A}^1(F)$. Thus evaluation at $w = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$, which is a T-homomorphism to $\eta^w \otimes \delta_B^{1/2}$ (which is equivalent to a B-homomorphism to $\mathrm{Ind}_T^B(\eta^w \otimes \delta_B^{1/2})$ by Frobenius reciprocity) lands in $c - \mathrm{Ind}_T^B(-)$. Can show that this map is an isomorphism. □

Remark 1.3.4. The induced B-map $V \to \mathrm{Ind}_T^B(\eta^w \otimes \delta_B^{1/2})$ is given by $f \mapsto (b \mapsto f(wb))$. Any $f \in V$ has support in an open compact subset K of $B\backslash BwB = B\backslash BwN$ and for any such open-compact subset there is an open-compact subgroup $N_0 \subset N$ such that f is supported on $B\backslash BwN_0$. From this it is clear that $b \mapsto f(wb) \in c - \mathrm{Ind}_T^B(\eta^w \otimes \delta_B^{1/2})$.

Proposition 1.3.5. *$J_B(V)$ is 1-dimensional and isomorphic to η^w.*

Proof. For $f \in V, x \in T$ let $f_N(x) = \int_N f(xwn)dn = \eta\delta_B^{-1/2}(x) f_N(1)$. Then the map $f \mapsto f_N(1)$ gives the isomorphism $V_N \cong \eta^w \otimes \delta_D^{1/2}$. □

Remark 1.3.6. This proof implicitly uses that the Haar measure on N is the restriction of the Haar measure on B, which follows from the theory of invariant measures.

Thus

$$0 \to V_N \to I(\eta)_N \to (\eta \otimes \delta_B^{-1/2})_N \to 0$$

is exact and we conclude that $J_B(I(\eta))$ is 2-dimensional and characters of T appearing are η and η^w.

Corollary 1.3.7. $\mathrm{Hom}_G(I(\eta), I(\eta'))$ *is* 0 *unless $\eta' = \eta$ or $\eta' = \eta^w$ and if $\eta \neq \eta^w$ in these cases it is* 1*-dimensional.*

Proof. We have

$$\mathrm{Hom}_G(I(\eta), I(\eta')) = \mathrm{Hom}_T(J_B(I(\eta)), \eta').$$

Let $\phi : J_B(I(\eta)) \to \eta'$ be a nontrivial T-homomorphism. Then ϕ is nontrivial on $\eta^w \subset J_B(I(\eta))$ if and only if $\eta^w = \eta'$ and if ϕ restricts to zero on η^w then it factors through the quotient $J_B(I(\eta))/\eta^w \cong \eta$ so this is non-zero if and only if $\eta' = \eta$. If $\eta \neq \eta^w$ then clearly only one of these cases can hold and so we conclude that in this case $\mathrm{Hom}_G(I(\eta), I(\eta'))$ is one-dimensional. □

Remark 1.3.8. 1. Slightly delicate to extract an explicit homomorphism $I(\eta) \to I(\eta^w)$.

2. Same proof shows that if θ is a non-trivial character of N, then $\dim\mathrm{Hom}_N(I(\eta), \theta) = 1$ ('uniqueness of Whittaker functionals').

3. Let V be as above and $W = \mathrm{Ker}(V \to V_N)$, a codimension 2 subrepresentation of $I(\eta)$

   Fact: W is irreducible as a B-representation

4. G/B is a G-variety with an open B-orbit (a spherical variety).

1.4 Supercuspidal representations

Recall that we say a smooth irreducible representation of $G = \mathrm{GL}_2(F)$ is *supercuspidal* if it is not a subspace of any $I(\chi, \psi)$.

Fact: If $p \neq 2$, there is an explicit description of supercuspidal representations.

Definition 1.4.1. An *admissible pair* is a pair (E, χ), where

- E/F is a quadratic extension
- $\chi : E^\times \to \mathbb{C}^\times$ is a smooth character

such that

1. χ doesn't factor through $\mathrm{Norm}_{E/F}$
2. If E is ramified then $\chi|_{1+\wp_E}$ doesn't factor through the norm.

Then there is a map

$$\{\text{admissible pairs } (E, \chi)\} \to \{\text{Supercuspidal } \mathrm{GL}_2(F)\text{-representations}\}$$

and if $p \neq 2$ all supercuspidal representations arise in this way.

2 Hecke algebras

2.1 Definitions

Let G be a locally profinite group, unimodular and let μ be a Haar measure on G. Let $C_c^\infty(G)$ be the space of locally constant compactly supported functions on G.

Definition 2.1.1. For $V \in \underline{Smo}_G$, $v \in V, F \in C_c^\infty(G)$ let

$$F * v = \int_G F(g) g \cdot v d\mu(g) \in V.$$

In particular, can take $V = C_c^\infty(G)$ with

$$(g \cdot F)(h) = F(g^{-1}h),$$

so

$$(F * G)(h) = \int_G F(g) G(g^{-1}h) d\mu(g).$$

Exercise: This is associative, so $C_c^\infty(G)$ becomes a ring $\mathcal{H}(G)$. and any $V \in \underline{Smo}_G$ an $\mathcal{H}(G)$-module.
Problem: $\mathcal{H}(G)$ has no identity element unless G discrete

Definition 2.1.2. Let $K \subset G$ be an open compact subgroup. Set

$$e_K = \frac{1}{\mu(K)} \mathbb{1}_K,$$

where $\mathbb{1}_K$ is the indicator function of K.

Now we have $e_K * e_K = e_K$ so the subring $\mathcal{H}(G, K) = e_K \mathcal{H}(G) e_K$ is unital, with identity element e_K.

For $V \in \underline{Smo}_G$, $e_K V = V^K$ and in particular, $\mathcal{H}(G, K)$ is the functions invariant under left and right translation by K.

Proposition 2.1.3. 1. *As a $\mathbb{C}$-vector space, $\mathcal{H}(G, K)$ has basis consisting of elements*

$$[KgK] = \frac{1}{\mu(K)} \mathbb{1}_{KgK}$$

for $g \in K \backslash G / K$.

2. *We have the formula*

$$[KgK][KhK] = \sum_{\gamma} c_\gamma [K\gamma K],$$

where c_γ is the integer

$$\mu(KgK \cap \gamma K h^{-1} K)/\mu(K)$$

which vanishes for all but finitely many γ.

Proof. First part is clear, second is an easy computation. □

If $V \in \underline{Smo}_G$ is irreducible, then V^K is a simple $\mathcal{H}(G, K)$-module or 0.

Theorem 2.1.4 (Bushnell and Henniart §4.3)**.** *This gives a bijection*

$$\{\textit{Irreducible } V \textit{ such that } V^K \neq 0\}/\cong \to \{\textit{Simple } \mathcal{H}(G, K)\textit{-modules}\}/\cong$$