Problem Set 1 – Solutions

Problem 1: Let $\{e_1, e_2, e_3\}$ be a $\mathbb{Z}$-basis for N. Let $\sigma = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_1, e_2, e_3\}$. Identify U_σ and U_τ for each face τ of σ.
Now consider the cone complex of these faces τ. Does this cone complex define a toric variety? If so, can you identify it? If not, why not?

Solution: $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ where $S_\sigma = \sigma^\vee \cap M$.

$\sigma = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_1, e_2, e_3\}$, so $\sigma^\vee = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_1^*, e_2^*, e_3^*\}$, and
$\mathbb{C}[S_\sigma] = \mathbb{C}[z^{e_1^*}, z^{e_2^*}, z^{e_3^*}] \cong \mathbb{C}[x_1, x_2, x_3]$.
Then $U_\sigma = \mathbb{C}^3$. (Again, when we write this we are indicating what we really mean by "$\mathbb{C}^3$". In reality $\mathbb{C}^3$ describes the closed points of U_σ.)
Now take $\tau_{ij} = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_i, e_j\}$. Then $\tau_{ij}^\vee = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_i^*, e_j^*, \pm e_k^*\}$, where $\{i, j, k\} = \{1, 2, 3\}$. $U_{\tau_{ij}} = \mathbb{C}^2_{x_i, x_j} \times \mathbb{C}^*_{x_k^{\pm 1}}$.

Next, take $\rho_i = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_i\}$. $\rho_i^\vee = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{e_i^*, \pm e_j^*, \pm e_k^*\}$, and
$U_{\rho_i} = \mathbb{C}_{x_i} \times (\mathbb{C}^*)^2_{x_j^{\pm 1}, x_k^{\pm 1}}$.
Finally, $\{0\}^\vee = M_{\mathbb{R}}$ and $U_{\{0\}} = T_N = (\mathbb{C}^*)^3_{x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}}$.

If we consider the cone complex of faces Σ of σ, the associated toric variety should be $X_\Sigma = \bigcup_{F \in \Sigma} U_F / \sim$.
Note that $X_\Sigma \subset U_\sigma$ since each U_F is, and - obtusely written - $X_\Sigma = U_\sigma \setminus (U_\sigma \setminus X_\Sigma)$. Now observe that $U_\sigma \setminus X_\Sigma = \{x_1 = x_2 = x_3 = 0\}$.
So $X_\Sigma = \mathbb{C}^3 \setminus \{0\}$.

Problem 2: Show that U_σ is normal.

Solution:

Lemma: If $V = \mathrm{Spec}(R)$ is an integral affine scheme, V is normal if and only if R is integrally closed.

First, it's clear that $\mathbb{C}[S_\sigma]$ is an integral domain — we have the stronger result that $\mathbb{C}[M]$ is. For any open $V \subset U_\sigma$, $\mathcal{O}(V) \subset \mathrm{Frac}(\mathbb{C}[S_\sigma])$, which is also clearly an integral domain. So U_σ is integral, and we are ready to use the lemma.

We need to see that $\mathbb{C}[S_\sigma]$ is integrally closed.

Note that $\sigma = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{\rho_i\}$, and $\sigma^\vee = \bigcap \rho_i^\vee$

So $S_\sigma = \bigcap_i S_{\rho_i}$, and $\mathbb{C}[S_\sigma] = \bigcap_i \mathbb{C}[S_{\rho_i}]$.

Then it's sufficient to show each $\mathbb{C}[S_{\rho_i}]$ is integrally closed. Let n_i be the primitive generator of ρ_i and complete to a $\mathbb{Z}$-basis $\{n_i, u_2, \ldots, u_d\}$ of N. Let $\{m_1, \ldots, m_d\}$ be the dual basis. Then $S_{\rho_i} = \mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{m_1, m_2^{\pm 1}, \ldots, m_d^{\pm 1}\}$ and $\mathbb{C}[S_{\rho_i}] = \mathbb{C}[x_1, x_2^{\pm}, \ldots, x_d^{\pm}]$. But $\mathbb{C}[x_1, \ldots, x_d]$ is integrally closed (it's a UFD)* so its localization is $\mathbb{C}[S_{\rho_i}]$ integrally closed as well.

* **UFD's integrally closed:** R a UFD, $x \in \mathrm{Frac}(R)$ integral over R.

Then we can write:

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0, \qquad a_i \in R.$$

Write $x = \frac{r}{s}$ with this expression reduced.

Then $r^n + s a_{n-1} r^{n-1} + \cdots + s^n a_0 = 0$ and s divides r^n. But in a UFD if $\frac{r}{s}$ is reduced and $s \mid r^n$ then s is a unit and so $r \in R$.

I'll discuss saturation at the beginning of the next lecture.

Problem 3: Problem 2 suggests a way to construct some non-normal toric varieties. Can you give such a construction?

Solution: One option is to take a non-saturated subsemigroup S of M.

For instance, we could still have a cone σ and take $S_\sigma = \sigma^\vee \cap M$, but specify some non-saturated $S \subset S_\sigma$. With this extra data, $U_S := \mathrm{Spec}(\mathbb{C}[S])$ becomes an affine non-normal toric variety. The inclusion $\mathbb{C}[S] \to \mathbb{C}[S_\sigma]$ induces the normalization morphism $U_\sigma \to U_S$.

Problem 4: Show that the cuspidal cubic plane curve $X = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x^3 - y^2\rangle)$ is non-normal. If you did Problem 3, try to use your construction to describe X.

Solution: We want to see that $R := \mathbb{C}[x,y]/\langle x^3 - y^2\rangle$ is **not** integrally closed in $K := \mathrm{Frac}(R)$. Consider $z := \frac{\bar{y}}{\bar{x}}$, where $\bar{x}$ and $\bar{y}$ are the classes of x and y in R. Then z is not in R since $\bar{x}$ is not a unit in R.

However, $z^2 - \bar{x} = 0$ since

$$z^2 = \left(\frac{\bar{y}}{\bar{x}}\right)^2 = \overline{\left(\frac{y^2}{x^2}\right)} = \overline{\left(\frac{x^3}{x^2}\right)} = \bar{x}.$$

z is integral over R, but not in R.

Let $\sigma = \mathbb{Z}_{\geq 0} \cdot e \subset \mathbb{Z} \cdot e = N$. Then $S_\sigma = \mathbb{Z}_{\geq 0} \cdot e^* \subset \mathbb{Z} \cdot e^* = M$. Now take $S = \mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{2e^*, 3e^*\}$. Then $\mathbb{C}[S] = \mathbb{C}[z^{2e^*}, z^{3e^*}] = \mathbb{C}[x,y]/\langle x^3 - y^2\rangle$.

$U_{S_\sigma} = \mathrm{Spec}(\mathbb{C}[z^{e^*}]) = \mathbb{C} \longrightarrow X = U_S$ is the normalization morphism.