1 Incompressible viscous fluid flow. The Navier-Stokes equations.

The purpose of this section is to give a brief summary of the Navier-Stokes equations for a flow of an incompressible viscous fluid.

1.1 Kinematic definitions.

The position vector of a fluid particle at time t is denoted as $\mathbf{r} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the usual Cartesian unit vectors. The velocity of a fluid particle is denoted as $\mathbf{u} = (u, v, w) = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Alternatively, we can write $\mathbf{r} = x_i\mathbf{e}_i$, $u = u_i\mathbf{e}_i$, where $i = 1, 2, 3$, $\mathbf{e}_i$ is the unit vector along the Cartesian axis x_i and the summation convention over repeated indices is used so that, for example,

$$x_i\mathbf{e}_i = \sum_{i=1}^{3} x_i\mathbf{e}_i. \tag{1.1}$$

There are two common approaches to describing the flow.

Lagrangian specification: all the relevant quantities (e.g. position vector, velocity, acceleration etc.) for a chosen fluid particle are functions of time and the position of the particle at some initial time, for instance at $t = 0$. For example, if $\mathbf{r}_0$ is the position vector of a fluid particle at time $t = 0$, then the position vector of the same particle at time t will be $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$.

Example 1. Given the position vector of a fluid particle in Lagrangian specification, its velocity and acceleration are calculated as follows:

$$\mathbf{u} = \frac{\partial \mathbf{r}(\mathbf{r}_0, t)}{\partial t}, \tag{1.2}$$

$$\mathbf{a} = \frac{\partial \mathbf{u}(\mathbf{r}_0, t)}{\partial t} = \frac{\partial^2 \mathbf{r}(\mathbf{r}_0, t)}{\partial t^2}. \tag{1.3}$$

Eulerian specification: each quantity related to the fluid flow is a function of time t and the position in space, $\mathbf{r} = (x, y, z)$. For instance, velocity and density in the flow are written as

$$\mathbf{u} = \mathbf{u}(\mathbf{r}, t) = \mathbf{u}(x, y, z, t), \qquad \rho = \rho(\mathbf{r}, t) = \rho(x, y, z, t), \tag{1.4}$$

respectively.

Example 2. Given the velocity field in Eulerian specification,

$$\mathbf{u} = \mathbf{u}(\mathbf{r}, t), \tag{1.5}$$

find the acceleration of the fluid particles in the flow.

We change to the Lagrange variables and use the results (1.2) and (1.3). Then

$$\mathbf{u} = \mathbf{u}(\mathbf{r}, t) = \mathbf{u}(\mathbf{r}(\mathbf{r}_0, t), t). \tag{1.6}$$

Also, for the acceleration,

$$\mathbf{a} = \frac{\partial \mathbf{u}(\mathbf{r}(\mathbf{r}_0, t), t)}{\partial t} = \frac{\partial \mathbf{u}(x(\mathbf{r}_0, t), y(\mathbf{r}_0, t), z(\mathbf{r}_0, t), t)}{\partial t} \tag{1.7}$$

$$= \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial x}\frac{\partial x(\mathbf{r}_0, t)}{\partial t} + \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial y}\frac{\partial y(\mathbf{r}_0, t)}{\partial t} + \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial z}\frac{\partial z(\mathbf{r}_0, t)}{\partial t} + \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} \tag{1.8}$$

$$= \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{1.9}$$

with

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{1.10}$$

This result is often written as

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} \tag{1.11}$$

where the differential operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \tag{1.12}$$

is the so-called *material* time derivative. In this context the partial time derivative in (1.12), $\partial/\partial t$, is called the *local* derivative with respect to time and the operator $(\mathbf{u} \cdot \nabla)$ is the *convective* derivative.

Example 3. Given the density and velocity distributions in the flow,

$$\rho = \rho(\mathbf{r}, t) \text{ and } \mathbf{u} = \mathbf{u}(\mathbf{r}, t), \tag{1.13}$$

find the rate of change of density in a fluid particle.

The answer is

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho. \tag{1.14}$$

1.2 Conservation of mass.

Conservation laws in a moving substance are conveniently formulated for a material volume, i.e. a volume containing the same fluid particles at all times. Consider a small material volume, δV. The mass in δV does not change,

$$\frac{D(\rho \delta V)}{Dt} = 0, \tag{1.15}$$

Note that

$$\frac{1}{\delta V}\frac{D(\rho \delta V)}{Dt} = \frac{D\rho}{Dt} + \rho\frac{1}{\delta V}\frac{D(\delta V)}{Dt}, \tag{1.16}$$

and, in coordinates,

$$\frac{1}{\delta V}\frac{D\left(\delta V\right)}{Dt} = \frac{1}{\delta x\delta y\delta z}\frac{D\left(\delta x\delta y\delta z\right)}{Dt} \tag{1.17}$$

$$= \frac{1}{\delta x}\frac{D\left(\delta x\right)}{Dt} + \frac{1}{\delta y}\frac{D\left(\delta y\right)}{Dt} + \frac{1}{\delta z}\frac{D\left(\delta z\right)}{Dt}. \tag{1.18}$$

Then, since

$$\frac{1}{\delta x}\frac{D\left(\delta x\right)}{Dt} = \frac{\delta u}{\delta x}, \tag{1.19}$$

with similar expressions for the other two terms in (1.18), we have, in the limit $\delta V \to 0$,

$$\frac{\delta u}{\delta x} \to \frac{\partial u}{\partial x}, \tag{1.20}$$

and therefore

$$\lim_{\delta V \to 0}\frac{1}{\delta V}\frac{D\left(\delta V\right)}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div}\mathbf{u}. \tag{1.21}$$

The limit, $\delta V \to 0$, applied to (1.15) with (1.16) yields

$$\frac{D\rho}{Dt} + \rho\operatorname{div}\mathbf{u} = 0. \tag{1.22}$$

The equivalent forms of (1.22) are

$$\frac{\partial\rho}{\partial t} + \left(\mathbf{u}\cdot\nabla\right)\rho + \rho\operatorname{div}\mathbf{u} = 0; \tag{1.23}$$

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\left(\rho\mathbf{u}\right) = 0. \tag{1.24}$$

The equations (1.22), (1.23), (1.24) are all different expressions of the mass conservation.

Fluid is called incompressible if

$$\frac{D\left(\delta V\right)}{Dt} = 0, \tag{1.25}$$

or

$$\operatorname{div}\mathbf{u} = 0. \tag{1.26}$$

The last equation is known as the continuity equation for an incompressible fluid. Then the equation of mass conservation becomes

$$\frac{\partial\rho}{\partial t} + \left(\mathbf{u}\cdot\nabla\right)\rho = 0. \tag{1.27}$$

The fluid is called homogeneous if

$$\rho \equiv \text{const}. \tag{1.28}$$

For a homogeneous incompressible flow the mass conservation equation is satisfied as long as the velocity field satisfies the continuity equation (1.26).

1.3 Rate of change of momentum in a material volume.

The task is to simplify the expression,

$$\frac{d}{dt}\int\limits_{V(t)}\rho\mathbf{u}dV. \tag{1.29}$$

Since mass in conserved in a small material volume , $\rho\delta V = const$, we have,

$$\frac{d}{dt}\int\limits_{V(t)}\rho\mathbf{u}dV = \int\limits_{V(t)}\rho\frac{D\mathbf{u}}{Dt}dV. \tag{1.30}$$

1.4 Motion near a fluid particle.

At a fixed moment in time, t, consider two particles at a small distance $\delta\mathbf{r}$ from each other.

Particle 1. Position vector $\mathbf{r} = (x_1, x_2, x_3)$, velocity $\mathbf{u}(\mathbf{r}) = (u_1, u_2, u_3)$.

Particle 2. Position vector $\mathbf{r} + \delta\mathbf{r}$, velocity $\mathbf{u}(\mathbf{r} + \delta\mathbf{r})$, with $\delta\mathbf{r} = (\delta x_1, \delta x_2, \delta x_3)$

Here we use Cartesian coordinates with the unit basis vectors $\mathbf{e}_i$, $i = 1, 2, 3$.

Our aim is to express the velocity of the second particle in terms of the velocity of the first particle and various spatial derivatives of the velocity field calculated at the point $\mathbf{r}$. Consider, for example, the x_1-component,

$$u_1(\mathbf{r} + \delta\mathbf{r}) = u_1(x_1 + \delta x_1, x_2 + \delta x_2, x_3 + \delta x_3) \tag{1.31}$$

$$= u_1(x_1, x_2, x_3) + \frac{\partial u_1(x_1, x_2, x_3)}{\partial x_1}\delta x_1 + \frac{\partial u_1(x_1, x_2, x_3)}{\partial x_2}\delta x_2 + \frac{\partial u_1(x_1, x_2, x_3)}{\partial x_3}\delta x_3 + O\left(|\delta\mathbf{r}|^2\right). \tag{1.32}$$

This result can be written shorter using the summation convention,

$$u_1(\mathbf{r} + \delta\mathbf{r}) = u_1(\mathbf{r}) + \frac{\partial u_1(\mathbf{r})}{\partial x_i}\delta x_i + \ldots\ . \tag{1.33}$$

Similar relations hold for $u_2(\mathbf{r} + \delta\mathbf{r})$, $u_3(\mathbf{r} + \delta\mathbf{r})$ and also for the complete vector velocity,

$$\mathbf{u}(\mathbf{r} + \delta\mathbf{r}) = u_j(\mathbf{r} + \delta\mathbf{r})\mathbf{e}_j \tag{1.34}$$

$$= \left[u_j(\mathbf{r}) + \frac{\partial u_j(\mathbf{r})}{\partial x_i}\delta x_i + \ldots\right]\mathbf{e}_j = u_j(\mathbf{r})\mathbf{e}_j + \frac{\partial u_j(\mathbf{r})}{\partial x_i}\delta x_i\mathbf{e}_j + \ldots \tag{1.35}$$

$$= \mathbf{u}(\mathbf{r}) + \frac{\partial u_j(\mathbf{r})}{\partial x_i}\delta x_i\mathbf{e}_j + \ldots = \mathbf{u}(\mathbf{r}) + \frac{\partial u_i(\mathbf{r})}{\partial x_j}\delta x_j\mathbf{e}_i + \ldots\ . \tag{1.36}$$

Note the change of subscripts in the last expression.

Using the identity,

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \tag{1.37}$$

and the notation,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{1.38}$$

we can re-write (1.36) as

$$\mathbf{u}\left(\mathbf{r} + \delta\mathbf{r}\right) = \underset{\text{translation}}{\mathbf{u}\left(\mathbf{r}\right)} + \underset{\text{deformation}}{e_{ij}\delta x_j \mathbf{e}_i} + \underset{\text{rotation}}{\xi_{ij}\delta x_j \mathbf{e}_i} + \dots \,. \tag{1.39}$$

We conclude that the local motion near a fluid particle can be described as a superposition of three motions: a pure translation with velocity $\mathbf{u}\left(\mathbf{r}\right)$, deformation characterized by the symmetric second-order tensor, e_{ij}, and rotation described by the antisymmetric second-order tensor, ξ_{ij}. The tensor e_{ij} is known as the rate-of-strain tensor.

1.5 Forces in fluids.

We distinguish between long-range or body forces (such as gravity) and short-range forces due to molecular interactions inside the fluid (pressure, friction). Another force that often appears in applications is surface tension (a short-range force acting on the boundary between two fluids).

According to Newton's Second Law, the rate of change of momentum of the fluid in a material volume equals the total force acting on this volume of fluid. The task for us is to learn how to describe the forces in the fluid. Long-range forces do not pose much problem. In the case of gravity, for example, the force on the fluid with density ρ contained in the volume $V\left(t\right)$ is

$$\int\limits_{V(t)} \rho \mathbf{g} dV, \tag{1.40}$$

where $\mathbf{g}$ is the acceleration due to gravity.

Short-range forces.

At a fixed moment in time, consider a volume of fluid with surface area A. Choose a small element of the surface area, δA, with the outward unit normal $\widehat{\mathbf{n}}$. Let $\delta\mathbf{F}$ be the force exerted on δA by the fluid outside of our volume.

For a small δA,

$$\delta\mathbf{F} \approx \mathbf{\Sigma}\delta A, \tag{1.41}$$

where $\mathbf{\Sigma}$ is the stress vector. Note that $\mathbf{\Sigma} = \mathbf{\Sigma}\left(\mathbf{r}, t, \widehat{\mathbf{n}}\right)$, i.e. the stress vector in general depends on the orientation of the area element.

We are looking to find a quantity which would be independent of the position of the surface area.

In Newton's Second Law,

$$\int\limits_{V} \rho \frac{D\mathbf{u}}{Dt} dV = \int\limits_{V} \rho \mathbf{g} dV + \int\limits_{A} \mathbf{\Sigma} dA, \tag{1.42}$$

let us assume that the typical linear size of the fluid volume, δr, say, is small. Then

$$\int_V \rho \frac{D\mathbf{u}}{Dt} dV = O\left(\delta r^3\right), \qquad \int_V \rho \mathbf{g} dV = O\left(\delta r^3\right), \qquad \int_A d\mathbf{F} = O\left(\delta r^2\right). \tag{1.43}$$

The balance of terms in (1.43) requires then

$$\int_A \mathbf{\Sigma} dA = 0. \tag{1.44}$$

From the last relation, it can be shown that the stress vector components are proportional to the unit normal, more precisely if we write $\mathbf{\Sigma} = \Sigma_i \mathbf{e}_i$ then the individual components Σ_i can be written in terms of a second-order stress tensor, σ_{ij}, as

$$\Sigma_i = \sigma_{ji} n_j \quad \text{or} \quad \Sigma_i = \sigma_{ij} n_j. \tag{1.45}$$

We have used here the fact (which follows from conservation of angular momentum) that the stress tensor is symmetric, $\sigma_{ij} = \sigma_{ji}$.

Physical meaning of the components of the stress tensor.

Consider a two-dimensional situation in which a surface element is aligned with the i-axis and normal to the j-axis. Then the i-component of the stress vector applied to this surface is

$$\Sigma_i = \sum_k \sigma_{ik} n_k = \left| \begin{array}{c} \text{note that} \\ n_i = 0, n_j = 1 \end{array} \right| = \sigma_{ij}. \tag{1.46}$$

in other words, σ_{ij} is the i-component of the force per unit area exerted on the surface normal to the j-axis. In particular, σ_{ij} with $i \neq j$ designates stress components tangential to the surface, whereas σ_{ii} stands for normal stresses.

1.6 Constitutive relation

This is a relation between the stress tensor and the rate-of-strain tensor. For an incompressible viscous fluid we assume

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \tag{1.47}$$

where δ_{ij} is the Kronecker tensor, p is pressure and μ is the viscosity coefficient. The constitutive relation (1.47) defines the Newtonian viscous fluid. In practice, the viscosity coefficient changes with temperature (recall cooking oil, for instance), however we take μ to be constant.

Incompressible inviscid fluid is defined by

$$\sigma_{ij} = -p\delta_{ij}, \tag{1.48}$$

in which case the stress tensor has non-zero elements on the main diagonal only.

Typical values for the viscosity coefficient μ and for the kinematic viscosity, $\nu = \mu/\rho$, for air, water and a very viscous substance such as glycerin are

$$\begin{array}{ccc} & \mu\left[\text{poise} = \frac{\text{gm}}{\text{cm}\cdot\text{sec}}\right] & \nu = \frac{\mu}{\rho}\left[\frac{\text{cm}^2}{\text{sec}}\right] \\ \text{Air at room temperature} & 0.00018 & 0.145 \\ \text{Water} & 0.0114 & 0.0114 \\ \text{Glycerin} & 23.3 & 18.5 \end{array} \tag{1.49}$$

1.7 The Navier-Stokes equations.

We assume that gravity is the only long-range force present and write Newton's momentum conservation equation for a material volume $V(t)$ bounded by the surface $A(t)$ as

$$\int_{V(t)} \rho \frac{D\mathbf{u}}{Dt} dV = \int_{V(t)} \rho \mathbf{g} dV + \int_{A(t)} \mathbf{\Sigma} dA. \tag{1.50}$$

In scalar components, using also

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla), \quad \Sigma_i = \sigma_{ij} n_j, \tag{1.51}$$

we have

$$\int_{V(t)} \rho \left[\frac{\partial u_i}{\partial t} + (\mathbf{u}\cdot\nabla) u_i\right] dV = \int_{V(t)} \rho g_i dV + \int_{A(t)} \sigma_{ij} n_j dA. \tag{1.52}$$

Using the Divergence Theorem,

$$\int_{A(t)} \sigma_{ij} n_j dA = \int_{V(t)} \frac{\partial \sigma_{ij}}{\partial x_j} dV, \tag{1.53}$$

hence

$$\rho \left[\frac{\partial u_i}{\partial t} + (\mathbf{u}\cdot\nabla) u_i\right] = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{1.54}$$

This relation is known as the Cauchy equation, applicable to any continuous medium.

Now we we shall specify the medium by taking the constitutive relation for a Newtonian viscous fluid,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}. \tag{1.55}$$

As a result,

$$\rho \left[\frac{\partial u_i}{\partial t} + (\mathbf{u}\cdot\nabla) u_i\right] = \rho g_i - \frac{\partial p}{\partial x_i} + 2\mu \frac{\partial e_{ij}}{\partial x_j}, \tag{1.56}$$

and, since

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.57}$$

$$\frac{\partial e_{ij}}{\partial x_j} = \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)\right) = \frac{1}{2}\nabla^2 u_i, \tag{1.58}$$

since, due to the continuity equation,

$$\frac{\partial u_j}{\partial x_j} = \mathrm{div}\mathbf{u} = 0. \tag{1.59}$$

The momentum equation (1.56) can now be written as

$$\rho\left[\frac{\partial u_i}{\partial t} + (\mathbf{u}\cdot\nabla)\, u_i\right] = \rho g_i - \frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i. \tag{1.60}$$

The vector equivalent of this equation, together with the continuity equation, forms the Navier-Stokes equations of motion for an incompressible viscous fluid,

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right] = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{u}, \tag{1.61}$$

$$\mathrm{div}\mathbf{u} = 0. \tag{1.62}$$

Comment. The gravity term can be incorporated into the pressure gradient term by defining the modified pressure,

$$p^* = p + \rho f, \tag{1.63}$$

where f is the gravitational potential, i.e. $\mathbf{g} = -\nabla f$. The momentum equation in (1.61) then takes the form

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right] = -\nabla p^* + \mu\nabla^2\mathbf{u}, \tag{1.64}$$

that is the effect of gravity in the momentum equation is absent. However, one should be careful with the changes in the boundary conditions if, for instance, an air/water boundary is present in the problem formulation.

1.8 Boundary conditions.

For a solid boundary, the velocity of the fluid must be equal to the velocity of the boundary. This is the no-slip condition adopted for a viscous fluid. In particular, if the boundary is at rest then

$$\mathbf{u} = 0 \text{ on the boundary.} \tag{1.65}$$

Recall that in an inviscid flow the corresponding boundary condition requires only the normal component of the fluid velocity to match the velocity of the solid boundary, allowing slip on the wall.

Special conditions are applied on interfaces between two immiscible fluids.

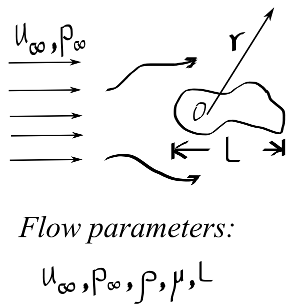

Figure 1: Flow past a body.

1.9 The Reynolds number.

Suppose a solid body is immersed in a uniform stream of incompressible viscous fluid. There are five parameters in the problem formulation: the fluid viscosity μ and density ρ, the speed and pressure in the incoming flow, u_∞ and p_∞, and the length parameter of the body, L. The Navier-Stokes equations of motion are

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right] = -\nabla p + \mu\nabla^2\mathbf{u}, \quad \nabla\cdot\ \mathbf{u} = 0. \tag{1.66}$$

The boundary conditions in the far field are

$$\mathbf{u} \to u_\infty \mathbf{i}, \quad p \to p_\infty \text{ as } \mathbf{r} \to \infty. \tag{1.67}$$

We also have no-slip conditions on the body surface,

$$\mathbf{u} = 0 \text{ if } \mathbf{r} \in \mathbf{\Gamma}, \tag{1.68}$$

where $\mathbf{\Gamma}$ denotes the body surface.

Let us reduce the number of parameters in the formulation. Define non-dimensional variables (denoted with an overbar)

$$\mathbf{r} = L\bar{\mathbf{r}}, \quad t = \frac{L}{u_\infty}\bar{t}, \quad \mathbf{u} = u_\infty\bar{\mathbf{u}}, \quad p = p_\infty + \rho u_\infty^2\bar{p}. \tag{1.69}$$

The differential operators change as follows,

$$\nabla = \frac{1}{L}\overline{\nabla}, \quad \nabla^2 = \frac{1}{L^2}\overline{\nabla}^2. \tag{1.70}$$

The problem formulation in non-dimensional variables becomes:

$$\frac{\partial \overline{\mathbf{u}}}{\partial \overline{t}} + \left(\overline{\mathbf{u}} \cdot \overline{\nabla}\right) \overline{\mathbf{u}} = -\overline{\nabla} \overline{p} + \frac{1}{Re} \overline{\nabla}^2 \overline{\mathbf{u}}, \quad \overline{\nabla} \cdot \ \overline{\mathbf{u}} = 0, \tag{1.71}$$

$$\overline{\mathbf{u}} \rightarrow \mathbf{i}, \quad \overline{p} \rightarrow 0 \text{ as } \overline{\mathbf{r}} \rightarrow \infty, \tag{1.72}$$

$$\overline{\mathbf{u}} = 0 \text{ if } \overline{\mathbf{r}} \in \overline{\mathbf{\Gamma}}. \tag{1.73}$$

The non-dimensional formulation contains just one parameter, the Reynolds number,

$$Re = \frac{\rho u_\infty L}{\mu} = \frac{u_\infty L}{\nu}. \tag{1.74}$$

1.10 Just one more calculation. Energy dissipation.

Consider the flow inside a sealed stationary container. The fluid occupies volume V bounded by the surface A.

The i-component of the Cauchy equation of motion is

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{1.75}$$

Here we ignore the effect of gravity and leave it for an exercise.

Multiplying (1.75) by u_i and taking the sum of the resulting three equations for $i = 1, 2, 3$, we have

$$\rho u_i \frac{\partial u_i}{\partial t} + \rho u_i u_j \frac{\partial u_i}{\partial x_j} = u_i \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{1.76}$$

Rearranging and using $u_i u_i = |\mathbf{u}|^2$, this can be written as

$$\frac{\rho}{2} \frac{\partial |\mathbf{u}|^2}{\partial t} + \rho \left[\frac{1}{2} \frac{\partial}{\partial x_j} \left(u_i u_i u_j \right) - \frac{1}{2} u_i u_i \underbrace{\frac{\partial u_j}{\partial x_j}}_{=0} \right] = \frac{\partial}{\partial x_j} \left(u_i \sigma_{ij} \right) - \sigma_{ij} \frac{\partial u_i}{\partial x_j}. \tag{1.77}$$

The term with div$\mathbf{u}$=0 on the left can be omitted. Integrating (1.77) over the volume of the container,

$$\frac{d}{dt} \int\limits_V \frac{1}{2} \rho |\mathbf{u}|^2 dV + \frac{\rho}{2} \underbrace{\int\limits_V \frac{\partial}{\partial x_j} \left(u_i u_i u_j \right) dV}_{I_1} = \underbrace{\int\limits_V \frac{\partial}{\partial x_j} \left(u_i \sigma_{ij} \right) dV}_{I_2} - \int\limits_V \sigma_{ij} \frac{\partial u_i}{\partial x_j} dV. \tag{1.78}$$

We note that the first integral,

$$K = \int\limits_V \frac{1}{2} \rho |\mathbf{u}|^2 dV, \tag{1.79}$$

represents the total kinetic energy of the fluid inside the container. The integrals $I_{1,2}$ both vanish on account of the Divergence Theorem and the no-slip condition, $u_i = 0, i = 1, 2, 3$, applied at the solid walls,

$$\begin{aligned} I_1 &= \int\limits_V \tfrac{\partial}{\partial x_j}\left(u_i u_i u_j\right) dV = \int\limits_A u_i u_i u_j n_j dA = 0, \\ I_2 &= \int\limits_V \tfrac{\partial}{\partial x_j}\left(u_i \sigma_{ij}\right) dV = \int\limits_A u_i \sigma_{ij} n_j = 0. \end{aligned} \tag{1.80}$$

As a result, we have the following energy equation,

$$\frac{dK}{dt} = -\int\limits_V \sigma_{ij} \frac{\partial u_i}{\partial x_j} dV. \tag{1.81}$$

This equation is valid for any continuous medium satisfying the Cauchy equation. For a Newtonian fluid, we specify the stress tensor,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \tag{1.82}$$

so that the energy equation becomes

$$\frac{dK}{dt} = \int\limits_V p\delta_{ij} \frac{\partial u_i}{\partial x_j} dV - 2\mu \int\limits_V e_{ij} \frac{\partial u_i}{\partial x_j} dV. \tag{1.83}$$

The integral representing the total work done by the pressure vanishes,

$$\int\limits_V p\delta_{ij} \frac{\partial u_i}{\partial x_j} dV = \int\limits_V p \frac{\partial u_j}{\partial x_j} dV = \int\limits_V p \mathrm{div} \mathbf{u} dV = 0, \tag{1.84}$$

on account of the continuity equation. For the second term on the right in (1.83) we have

$$e_{ij} \frac{\partial u_i}{\partial x_j} = \frac{1}{2} e_{ij} \frac{\partial u_i}{\partial x_j} + \underbrace{\frac{1}{2} e_{ij} \frac{\partial u_i}{\partial x_j}}_{\text{swap i and j}} = \frac{1}{2} e_{ij} \frac{\partial u_i}{\partial x_j} + \frac{1}{2} e_{ji} \frac{\partial u_j}{\partial x_i} = \left| \begin{array}{c} \text{recall} \\ e_{ij} = e_{ji} \end{array} \right| \tag{1.85}$$

$$= \frac{1}{2} e_{ij} \frac{\partial u_i}{\partial x_j} + \frac{1}{2} e_{ij} \frac{\partial u_j}{\partial x_i} = e_{ij} e_{ij}, \tag{1.86}$$

since

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.87}$$

Substituting (1.86) into (1.83), we arrive at the final energy equation,

$$\frac{dK}{dt} = -2\mu \int\limits_V e_{ij} e_{ij} dV. \tag{1.88}$$

Since $e_{ij}e_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij}^2$ is a positive definite function, the equation (1.88) shows that the kinetic energy in the flow is a monotonically decaying function of time. The function,

$$\Phi(\mathbf{r}, t) = 2\mu e_{ij}e_{ij}, \tag{1.89}$$

is referred to as the dissipation function and represents the rate of energy dissipation per unit volume.

EXERCISES.

Exercise 1. Let J be the intensity of internal sources of fluid, that is the rate of fluid production such that the formula (1.15) is replaced by

$$\frac{D(\rho\delta V)}{Dt} = J\delta V. \tag{1.90}$$

Derive the mass conservation equation in the form

$$\frac{D\rho}{Dt} + \rho \mathrm{div}\mathbf{u} = J. \tag{1.91}$$

Exercise 2. Verify the product rule,

$$\frac{D(\mathbf{a}\cdot\mathbf{b})}{Dt} = \frac{D\mathbf{a}}{Dt}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{D\mathbf{b}}{Dt}. \tag{1.92}$$

Exercise 3. Verify the formula,

$$\frac{d}{dt}\int\limits_{V(t)} \rho\mathbf{u}dV = \int\limits_{V(t)} \left[\rho\frac{D\mathbf{u}}{Dt} + J\mathbf{u}\right] dV, \tag{1.93}$$

for the rate of change of momentum in a material volume in the presence of internal sources of fluid of intensity J. This is a generalization of (1.30).

Exercise 4. What is more viscous - air or water?

Exercise 5. Show that the equation (1.88) holds for the flow with gravity,

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \rho g_i + \frac{\partial\sigma_{ij}}{\partial x_j}. \tag{1.94}$$

Hint. Introduce the modified pressure.

Exercise 6. Does the flow of a Newtonian viscous fluid terminate at a finite time due to frictional forces, or continues forever?