2 Matched asymptotic expansions.

In this section we develop a technique used in solving problems which exhibit boundary-layer phenomena. The technique is known as method of matched asymptotic expansions which we shall demonstrate on simple examples. First, we shall try to understand the nature of the singularity in the exact solution of one boundary-value problem for a second-order ordinary differential equation.

2.1 Exact solution of a boundary-value problem.

Example. Solve the following boundary value problem for the function, $y = y(x;\varepsilon)$, with $\varepsilon > 0$,

$$\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0,\ y(0;\varepsilon) = 0,\ y(1;\varepsilon) = 1. \tag{2.1}$$

In the exact solution, by writing $y = e^{\lambda x}$, we first find the characteristic equation,

$$\varepsilon\lambda^2 + \lambda - 1 = 0, \tag{2.2}$$

with the solution for the roots of the quadratic,

$$\lambda_\pm = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}, \tag{2.3}$$

and hence write the general solution as

$$y = C_1 e^{\lambda_+ x} + C_2 e^{\lambda_- x}. \tag{2.4}$$

Using the boundary conditions to find the undetermined constants we have the equations,

$$C_1 + C_2 = 0, \tag{2.5}$$

$$C_1 e^{\lambda_+} + C_2 e^{\lambda_- x} = 1, \tag{2.6}$$

hence $C_1 = (e^{\lambda_+} - e^{\lambda_-})^{-1}$, $C_2 = -(e^{\lambda_+} - e^{\lambda_-})^{-1}$, giving the required exact solution,

$$y(x;\varepsilon) = \frac{1}{e^{\lambda_+} - e^{\lambda_-}} \left(e^{\lambda_+ x} - e^{\lambda_- x}\right). \tag{2.7}$$

Solution curves for several values of ε are shown in Figure 1.

From Figure 1 we conclude that the solution develops on two different x-scales for small values of ε, namely in the region $x = O(1)$ and, in addition, in a small $x-$range near $x = 0$, with this narrow region shrinking to zero as $\varepsilon \to 0$. We can verify these properties analytically by approximating the solution in the limit $\varepsilon \to 0$.

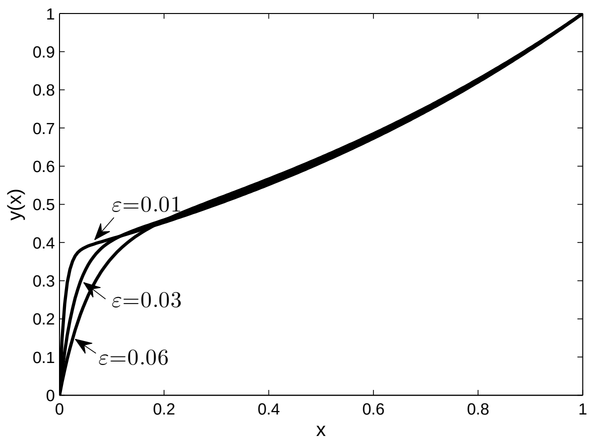

Figure 1: Exact solutions.

2.2 Approximation of the exact solution for small values of ε.

At small values of ε the roots of the characteristic equation can be expanded as

$$\lambda_+ = 1 - \varepsilon + O(\varepsilon^2), \tag{2.8}$$

$$\lambda_- = -\frac{1}{\varepsilon} - 1 + \varepsilon + O(\varepsilon^2), \tag{2.9}$$

with the difference in the order of magnitude of the two roots indicating two different x-scales in the solution.

Outer expansion.

In the limit $\varepsilon \to 0$, with x value fixed, $x \neq 0$, we have

$$y_{outer} = y(x; \varepsilon) = e^{x-1} + \varepsilon(1-x)e^{x-1} + O(\varepsilon^2). \tag{2.10}$$

We notice that the boundary condition, $y = 1$ at $x = 1$, is satisfied at order one and order ε (and can be shown to hold at higher orders in ε) however the boundary condition at $x = 0$ is violated in this expansion. The term 'outer' attached to this straightforward approximation valid in the majority of the x-range is used traditionally and implies the need for an alternative, 'inner', expansion in such situations.

Inner expansion.

An alternative expansion is needed for small values of x, specifically when $x = \varepsilon z$, with $z = O(1)$ as $\varepsilon \to 0$. This can be seen from the form of the exponential with λ_-in the full solution (2.7).

We have

$$y_{inner} = y(\varepsilon z; \varepsilon) = e^{-1}\left(1 - e^{-z}\right) + \varepsilon e^{-1}\left[z\left(1 + e^{-z}\right) + 1 - e^{-z}\right] + O\left(\varepsilon^2\right). \tag{2.11}$$

We observe that in the second, inner, expansion the boundary condition is satisfied at $x = 0$ (or $z = 0$) but not at $z = 0$.

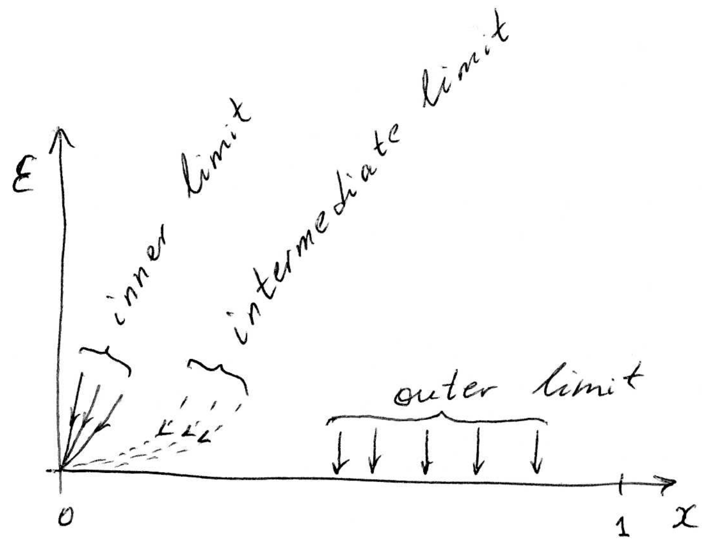

Figure 2: Outer, inner and intermediate limits in the (x, ε)-plane.

The two limits are illustrated in Figure 2.

Intermediate limits.

A question to ask now is whether the two limits, outer and inner, are sufficient to describe the singularity in the solution. We check this by taking intermediate limits, $\zeta = x/\varepsilon^\alpha = O(1)$ as $\varepsilon \to 0$ with $0 < \alpha < 1$. From the full solution (2.7), we have

$$y_{interm} = y(\varepsilon^\alpha \zeta, \varepsilon) = e^{-1} + \varepsilon^\alpha e^{-1}\zeta + O\left(\varepsilon^{2\alpha}, \varepsilon\right), \tag{2.12}$$

where we keep just the first two terms of the expansion.

It may come as a surprise, but the intermediate expansion (2.12) is, in fact, contained in both the outer and inner limits. Indeed, if we re-write the outer expansion (2.10) in the intermediate variable we have, using substitution $x = \varepsilon^\alpha \zeta$,

$$(y_{outer})_{interm} = e^{\varepsilon^\alpha \zeta - 1} + \varepsilon(1 - \varepsilon^\alpha \zeta)e^{\varepsilon^\alpha \zeta - 1} + O(\varepsilon^2) = e^{-1} + \varepsilon^\alpha e^{-1}\zeta + O\left(\varepsilon^{2\alpha}, \varepsilon\right). \tag{2.13}$$

Next, using $z = \varepsilon^{\alpha-1}\zeta$, we can re-write the inner expansion (2.11) in terms of the same intermediate variable,

$$(y_{inner})_{interm} = e^{-1}\left(1 - e^{-\varepsilon^{\alpha-1}\zeta}\right) + \varepsilon e^{-1}\left[\varepsilon^{\alpha-1}\zeta\left(1 + e^{-\varepsilon^{\alpha-1}\zeta}\right) + 1 - e^{-z}\right] + O\left(\varepsilon^2\right) \tag{2.14}$$

$$= e^{-1} + \varepsilon^\alpha e^{-1}\zeta + O\left(\varepsilon^{2\alpha}, \varepsilon\right). \tag{2.15}$$

From this analysis we can draw three conclusions.

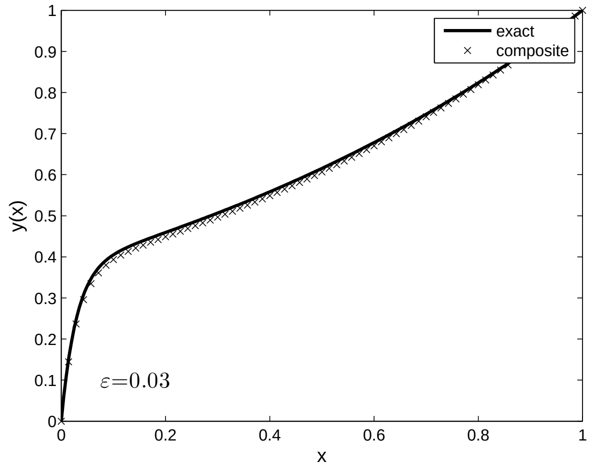

Figure 3: Exact solution (solid) vs composite expansion uniformly valid at leading order (crosses).

(i) Two asymptotic expansions, outer and inner, are sufficient to describe the solution for small values of ε in the entire x-range.

(ii) Whatever method we choose to construct outer and inner expansions, these two expansions must be identical in the intermediate limit. This, in fact, constitutes the main principle of matching when working with asymptotic expansions.

(iii) Since the outer and inner expansions overlap in the intermediate limit, a uniformly valid, composite, approximation to the function $y = y(x;\varepsilon)$ can be constructed following the rule

$$y_{comp}(x;\varepsilon) = y_{outer} + y_{inner} - y_{interm}. \tag{2.16}$$

Application of this rule to our expansions gives, to leading order,

$$y_{comp}(x;\varepsilon) = e^{x-1} + e^{-1}\left(1 - e^{-z}\right) - e^{-1} + ..., \tag{2.17}$$

or, when re-written in the original variable x,

$$y_{comp}(x;\varepsilon) = e^{x-1} + e^{-1}\left(1 - e^{-x/\varepsilon}\right) - e^{-1} + ... \tag{2.18}$$

A comparison between the exact and composite asymptotic solutions is shown in Figure 3.

Note. We often describe the various limits and expansions (outer, inner, intermediate) as regions on the $x-$axis. So the limit $x = O(1)$ with $\varepsilon \to 0$ is interpreted as the majority of the $x-$range excluding the origin, the limit $x = \varepsilon z$, $z = O(1)$, $\varepsilon \to 0$ corresponds to a region of length 'of order ε' near the origin, the intermediate limit becomes a somewhat wider region near the origin of length $O(\varepsilon^\alpha)$. If nothing else, such terminology reflects a convenient geometric imagery behind limit processes. See e.g. Hinch or Nayfeh's books for more examples.

2.3 Solution of the boundary-value problem by matched asymptotic expansions.

If the exact solution is not available we may still be able to construct an approximation of the solution using inner and outer asymptotic expansions. In this subsection we show how the method applies to the problem (2.1).

Outer expansion. We begin with a straightforward expansion in powers of ε as seems natural for values of x of $O(1)$. Hence

$$y(x;\varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + O(\varepsilon^3). \tag{2.19}$$

Substitution into equation (2.1) gives

$$\varepsilon y_0'' + \varepsilon^2 y_1'' ... + y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + ... - (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + ...) = 0, \tag{2.20}$$

where the dots denote terms of order $\varepsilon^3$and higher. This can be re-arranged collecting terms with similar powers of ε,

$$(y_0' - y_0) + \varepsilon(y_1' - y_1 + y_0'') + \varepsilon^2(y_2' - y_2 + y_1'') + O(\varepsilon^3) = 0. \tag{2.21}$$

Coefficients to powers of ε in (2.21) must vanish as can be seen from the following procedure. Taking the limit, $\varepsilon \to 0$, in the equation (2.21) results in

$$y_0' - y_0 = 0. \tag{2.22}$$

Setting the first bracket in (2.21) to zero and diving the equation by ε we have,

$$(y_1' - y_1 + y_0'') + \varepsilon(y_2' - y_2 + y_1'') + O(\varepsilon^2) = 0. \tag{2.23}$$

Taking the limit $\varepsilon \to 0$ in this new equation gives us

$$y_1' - y_1 + y_0'' = 0, \tag{2.24}$$

with the procedure repeated to find, at the next order,

$$y_2' - y_2 + y_1'' = 0. \tag{2.25}$$

As we can see, a sequence of equations emerges, first to determine y_0 from(2.22), then y_1 from (2.24), y_2 from (2.25) and so on, continuing to higher powers of ε if needed.

Solving (2.22) we find,

$$y_0(x) = C_0 e^x, \tag{2.26}$$

with a constant C_0 which should be determined from the boundary condition. This is not a straightforward task, however, for the starting formulation, (2.1), contains two boundary conditions. At this point we should acknowledge the singular nature of the perturbation problem and decide which of the two end conditions needs to be abandoned. Let us *assume*, that a boundary-layer expansion exists near $x = 0$ as a complement to the outer expansion above. Then the boundary condition, $y = 1$ at $x = 1$,gives

$$y_0(1) + \varepsilon y_1(1) + \varepsilon^2 y_2(1) + O(\varepsilon^2) = 1, \tag{2.27}$$

leading to the sequence of end conditions,

$$y_0(1) = 1, \tag{2.28}$$

$$y_1(1) = 0, \tag{2.29}$$

$$y_2(1) = 0, \tag{2.30}$$

for the consecutive terms in the outer expansion. Hence we find, using (2.28) in (2.26), that $C_0 = e^{-1}$. At the next order in ε, solving (2.24) with (2.29), we determine y_1,namely

$$y_1(x) = (1 - x)e^{x-1}. \tag{2.31}$$

Leaving it as an exercise to calculate the next term, $y_2(x)$, as a solution of (2.25) with (2.30), we can now write down the first terms in the outer expansion,

$$y(x; \varepsilon) = e^{x-1} + \varepsilon(1 - x)e^{x-1} + O(\varepsilon^2). \tag{2.32}$$

Inner limit. Distinguished scaling. First we need to determine appropriate scaling for the independent variable in the inner expansion valid when $x \to 0$ as $\varepsilon \to 0$. Let

$$x = \delta(\varepsilon)z,\ z = O(1), \tag{2.33}$$

where $z = O(1)$ is the new independent variable in the inner limit and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Substitution into the governing equation in (2.1) yields

$$\frac{d^2y}{dz^2} + \frac{\delta}{\varepsilon}\frac{dy}{dz} - \frac{\delta^2}{\varepsilon}y = 0. \tag{2.34}$$

Depending on a relation between the two small parameters, $\delta$and ε, the limiting form of the equation (2.34) can vary, however the least degenerate form is obtained with $\delta(\varepsilon) = \varepsilon$, as can be verified by trial. Then, in the limit, we have

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} = 0. \tag{2.35}$$

We observe that, as a result of the scaling transformation for the independent variable, the highest derivative is recovered in the governing equation, as required by the second-order nature of the original problem.

The scaling of the independent variable giving rise to the least degenerate form of the governing equation is known as a distinguished scaling and the limiting form of the equation as a distinguished limit of the equation. Finding distinguished limits is an important step in constructing matched asymptotic expansions.

Inner expansion. Once the scaling for the inner expansion is established, the formal calculation becomes relatively straightforward. With $\delta(\varepsilon) = \varepsilon$,the equation (2.34) becomes

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} - \varepsilon y = 0. \tag{2.36}$$

Solution is sought in the form,

$$y(z;\varepsilon) = Y_0(z) + \varepsilon Y_1(z) + O(\varepsilon^2). \tag{2.37}$$

From (2.36) and the boundary condition, $y = 0$ at $x = 0$, which we need to write now as $y = 0$ at $z = 0$, we derive a sequence of equations, starting with the leading order,

$$Y_0'' + Y_0' = 0,\ Y_0(0) = 0, \tag{2.38}$$

and then at the next order,

$$Y_1'' + Y_1' - Y_0 = 0,\ Y_1(0) = 0, \tag{2.39}$$

and so on.

Solving (2.38) for the leading term, we find

$$Y_0(z) = K_0(1 - e^{-z}), \tag{2.40}$$

with one constant, K_0, still undetermined at this stage. Note that it would be wrong to impose the boundary condition at $x = 1$ to the solution in the inner region since the the point $x = 1$ is outside the region of validity of the inner expansion.

At the next order, solving (2.39),

$$Y_1(z) = K_0 z(1 + e^{-z}) + K_1(1 - e^{-z}), \tag{2.41}$$

with another undetermined constant, K_1. Using (2.37), we now have the first two terms of the inner expansion,

$$y = K_0(1 - e^{-z}) + \varepsilon \left[K_0 z(1 + e^{-z}) + K_1(1 - e^{-z})\right] + O(\varepsilon^2). \tag{2.42}$$

Intermediate matching. The outer expansion (2.32) and inner expansion (2.42) must coincide in the intermediate limit. Let

$$x = \varepsilon^\alpha \zeta \text{ with } \zeta = O(1)\text{and } 0 < \alpha < 1. \tag{2.43}$$

In terms of the inner variable we have $z = \varepsilon^{\alpha-1}\zeta$. The intermediate limit of the inner expansion (2.42) is

$$y = K_0 + \varepsilon^\alpha K_0 \zeta + \varepsilon K_1 + ..., \tag{2.44}$$

where we have only kept the terms of order ε and greater. The intermediate limit if the outer expansion (2.32) is obtained as follows,

$$y = e^{x-1} + \varepsilon(1 - x)e^{x-1} + O(\varepsilon^2) \tag{2.45}$$

$$= e^{-1}e^{\varepsilon^\alpha\zeta} + \varepsilon(1-\varepsilon^\alpha\zeta)e^{-1}e^{\varepsilon^\alpha\zeta} + O(\varepsilon^2) \tag{2.46}$$

$$= e^{-1}(1+\varepsilon^\alpha\zeta + O(\varepsilon^{2\alpha})) + \varepsilon(1-\varepsilon^\alpha\zeta)e^{-1}(1+\varepsilon^\alpha\zeta + O(\varepsilon^{2\alpha})) + O(\varepsilon^2) \tag{2.47}$$

$$= e^{-1} + \varepsilon^\alpha\zeta e^{-1} + \varepsilon e^{-1} + O(\varepsilon^{2\alpha}, \varepsilon^{1+2\alpha}, \varepsilon^2). \tag{2.48}$$

Comparing (2.44) and (2.48) we determine the constants,

$$K_0 = e^{-1},\ K_1 = e^{-1}. \tag{2.49}$$

This, essentially, completes the solution of the boundary-value problem in the first two orders of approximation.

For discussion. Van Dyke's matching principle.

EXERCISES.

Exercise 1. Find a uniformly valid, composite, solution of the boundary-value problem discussed above using the inner and outer expansions derived already.

Exercise 2. The intermediate expansion of the outer solution (2.49) contains terms of order $\varepsilon^{2\alpha}$ and $\varepsilon^{1+2\alpha}$. Do we have to match these terms with something in the inner expansion? Also, related to this, how can we be certain that the term of $O(\varepsilon^2)$in the outer expansion remains of order $O(\varepsilon^2)$ in the intermediate region?

Exercise 3. Compare the following two boundary-value problems for a first-order differential equation:

$$(i)\ \ \varepsilon\frac{dy}{dx} + y = 1 - x,\ \ x \in [0,1],\ \ y(x=0;\varepsilon) = 0. \tag{2.50}$$

$$(ii)\ \ \varepsilon\frac{dy}{dx} + y = 1 - x,\ \ x \in [0,1],\ \ y(x=1;\varepsilon) = 1. \tag{2.51}$$

In both cases, attempt an approximate solution using matched asymptotic expansions and see what happens. If you get stuck refer to an exact solution.

Exercise 4. How do we decide on the location of the boundary layer (near $x = 0$ or near $x = 1$) in constructing the inner expansion?

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