

## PROBLEMS 2

*The Bivariate Normal Distribution.* Define

$$f(x, y) = c \exp\left\{-\frac{1}{2}Q(x, y)\right\},$$

where  $c$  is a constant,  $Q$  a positive definite quadratic form in  $x$  and  $y$ . Specifically:

$$c = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \quad Q = \frac{1}{1-\rho^2} \left[ \left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 \right].$$

Here  $\sigma_i > 0$ ,  $\mu_i$  are real,  $-1 < \rho < 1$ . Show that:

- Q1.  $f$  is a probability density – that is, that  $f$  is non-negative and integrates to 1.  
 Q2. If  $f$  is the density of a random 2-vector  $(X, Y)$ ,  $X$  and  $Y$  are normal, with distributions  $N(\mu_1, \sigma_1^2)$ ,  $N(\mu_2, \sigma_2^2)$ .  
 Q3.  $X, Y$  have means  $\mu_1, \mu_2$  and variances  $\sigma_1^2, \sigma_2^2$ .  
 Q4. The conditional distribution of  $y$  given  $X = x$  is  $N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2))$ .  
 Q5. The conditional mean  $E(Y|X = x)$  is *linear* in  $x$ :

$$E(Y|X = x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1).$$

- Q6.  $\text{var}[Y|X] = \sigma_2^2(1 - \rho^2)$ .  
 Q7. The correlation coefficient of  $X, Y$  is  $\rho$ .  
 Q8. The density  $f$  has elliptical contours [i.e., the curves  $f(x, y)$  constant are ellipses].  
 Q9. The joint MGF and joint CF of  $X, Y$  are

$$M_{X,Y}(t_1, t_2) = M(t_1, t_2) = \exp(\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}[\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2]),$$

$$\phi_{X,Y}(t_1, t_2) = \phi(t_1, t_2) = \exp(i\mu_1 t_1 + i\mu_2 t_2 - \frac{1}{2}[\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2]).$$

- Q10.  $X, Y$  are independent if and only if  $\rho = 0$ .

*Note.* For those of you with a background in Statistics, this will be familiar material. It is included here as it serves as a very concrete illustration of the more abstract conditioning of Week 2 via the Radon-Nikodym Theorem. For those of you without a background in Statistics: the key here is *completing the square* (the method you first encountered in learning how to solve quadratic equations). If you need help, find a good textbook on Statistics (e.g., [BF]) and look up ‘bivariate normal distribution’ in the index. NHB